Bayesian Inference

"The true logic for this world is the calculus of Probabilities, which takes account of the magnitude of the probability which is, or ought to be, in a reasonable man's mind."

James Clerk Maxwell *(1850)*

Jérémie Mattout

Lyon Neuroscience Research Center, France

With many thanks to

Jean Daunizeau

Guillaume Flandin

Karl Friston

Will Penny

SPM EEG-MEG Course

Outline

- General principles

- The Bayesian way

- SPM examples

- **General principles**
- The Bayesian way
- SPM examples

A starting point

Statistics: concerned with the collection, analysis and interpretation of data to make decisions

Applied statistics

Theoretical statistics

Descriptive statistics

summary statistics, graphics…

Probability

Inferential statistics

Data interpretations, decision making

(Modeling, accounting for randomness and unvertainty, hypothesis testing, infering hidden parameters)

The notion(s) of probability

B. Pascal (1623-1662)

P. de Fermat (1601-1665)

To express belief that an event has or will occur

Ω *: All possible events*

A_i *: one particular event*

Kolomogorov axioms

A.N. Kolmogorov (1903-1987)

(1) $0 \leq P(A) \leq 1$

(2) $P(\Omega) = 1$

(3) $P(A_1 \cup A_2 \cdots \cup A_k) = \sum_{i=1}^{k} P(A_i)$

(for mutually exclusive events)

A few consequences…

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(joint probability)

$$P(A \cap B) = 0$$

(if mutually exclusive events)

$$P(A \cap B) = P(A).P(B)$$

(if independent events)

The notion(s) of probability

Frequentist interpretation

- ***Probability*** *= frequency of the occurrence of an event, given an infinite number of trials*

- *Is only defined for random processes that can be observed many times*

- *Is meant to be* ***Objective***

Bayesian interpretation

- ***Probability*** *= degree of belief, measure of uncertainty*

- *Can be arbitrarily defined for any type of event*

- *Is considered as* ***Subjective*** *in essence*

The notion(s) of probability

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Joint and conditional probabilities

- *Joint probability of A and B* $P(A \cap B) = P(A, B)$
- *Conditional probability of A given B* $P(A|B)$

$$P(A, B) = P(A|B)P(B)$$

- *Note that if A and B are independent*

$$P(A|B) = P(A)$$

and

$$P(A, B) = P(A)P(B)$$

Joint and conditional probabilities

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$$P(A, B) = P(A|B)P(B)$$

$$P(A, B) = P(B, A) = P(B|A)P(A)$$

$$P(A|B)P(B) = P(B|A)P(A)$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

T. Bayes (1702-1761)

Extension to multiple variables

$$P(A,B,C) = P(A,B|C)P(C) = P(A|B,C)P(B|C)P(C)$$
$$= P(B|A,C)P(A|C)P(C)$$

$$P(A|B,C) = \frac{P(B|A,C)P(A|C)}{P(B|C)}$$

T. Bayes (1702-1761)

Marginalisation

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- Discrete case

$$P(B) = \sum_A P(A, B) = \sum_A P(B|A)P(A)$$

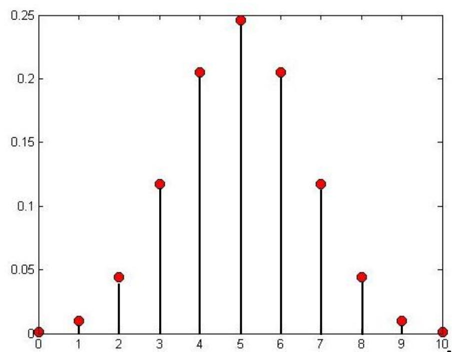
- Continuous case

$$P(B) = \int P(A, B)dA = \int P(B|A)P(A)dA$$

Probability distributions (quick reminder)

Discrete variable
(e.g. Binomial distribution)

$$P(Heads) = 1 - P(Tails)$$



Number of Heads after 10 trials

$$p(X = x) = C_x^n p^x (1-p)^{n-x}$$

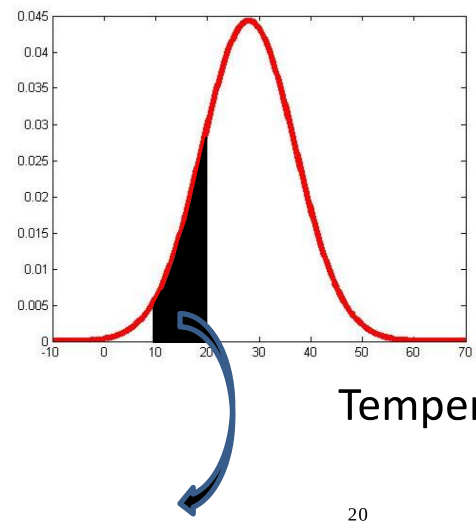
$$p(X \leq x) = \sum_{0}^{x} f(x)$$

Continuous variable
(e.g. Gaussian distribution)

$$p(X) \sim N(\mu, \sigma)$$

$$p(X = x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$p(X)$



Temperature *X*

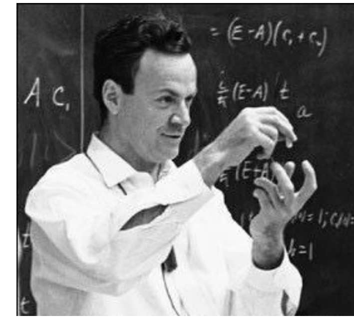
$$p(10 \leq X \leq 20) = \int_{x=10}^{20} f(x)dx$$

- General principles
- **The Bayesian way**
- SPM examples

A word on generative models

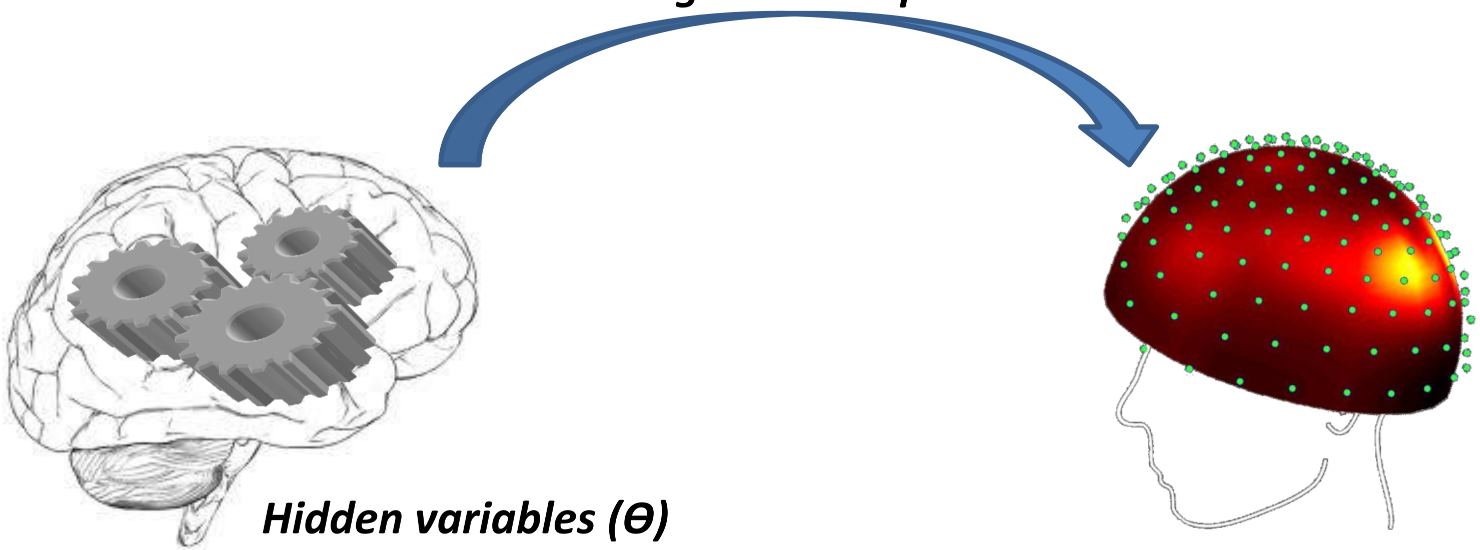
What I cannot create, I do not understand.

Richard Feynman (1918 – 1988)



<u>Model:</u> mathematical formulation of a system or process (set of hypothesis and approximations)

Data generative process ?



Observations (Y)

<u>A Probabilistic Model enables to:</u>

- **Account for prior knowledge and uncertainty**
(due to randomness, noise, incomplete observations)
- Simulate data
- Make predictions
- **Estimate hidden parameters**
- **Test Hypothesis**

Another look at Bayes rule

Model/Hypothesis

Likelihood

Prior

$$P(\theta|Y, M) = \frac{P(Y|\theta, M)P(\theta|M)}{P(Y|M)}$$

Posterior
or conditional

Marginal likelihood
or evidence

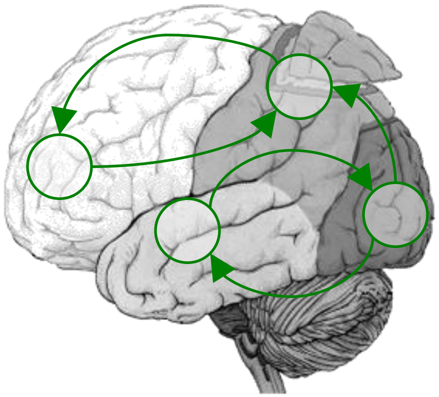
To be infered

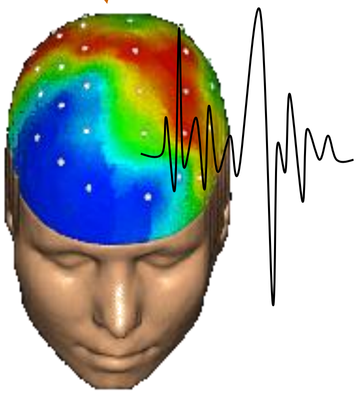
Another look at Bayes rule

forward problem

$$P\left(Y \middle| \theta, M\right)$$

likelihood





posterior distribution

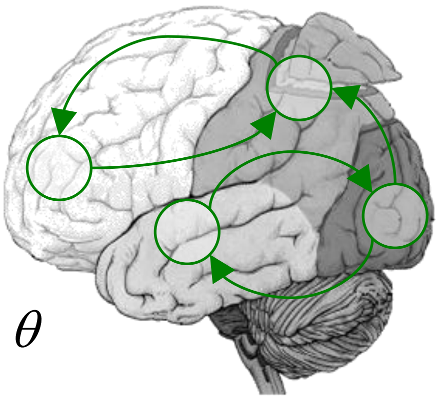
$$P\left(\theta \middle| Y, M\right)$$

inverse problem

Likelihood function

$$P(\theta|Y,M) = \frac{P(Y|\theta,M)P(\theta|M)}{P(Y|M)}$$

Assumption $Y = f(\theta)$



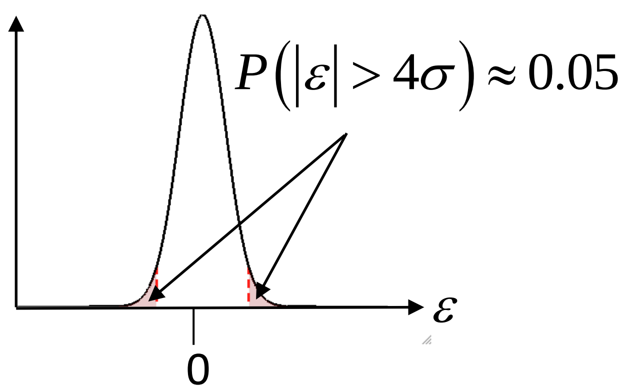
θ

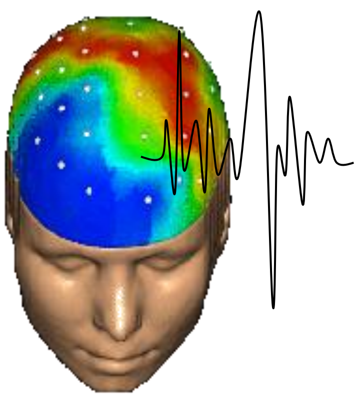
e.g. linear model $Y = X\theta$

But data are noisy $Y = X\theta + \varepsilon$

f

$$p(\varepsilon) \propto \exp\left(-\frac{1}{2\sigma^2}\varepsilon^2\right)$$





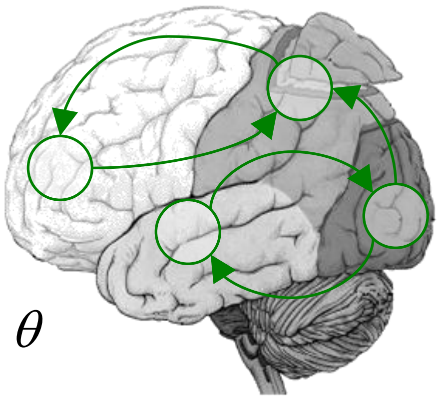
Y

Distribution of data, *given fixed parameters*:

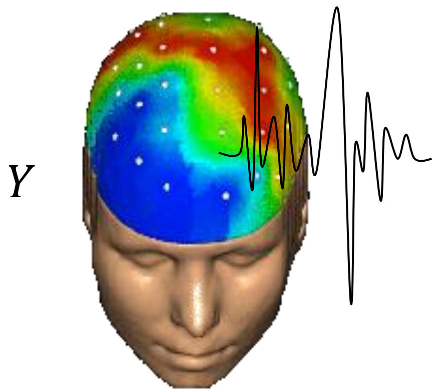
$$p(y|\theta) \propto \exp\left(-\frac{1}{2\sigma^2}(y - f(\theta))^2\right)$$

Adding priors: a simple example

$$P(\theta|Y,M)=\frac{P(Y|\theta,M)P(\theta|M)}{P(Y|M)}$$

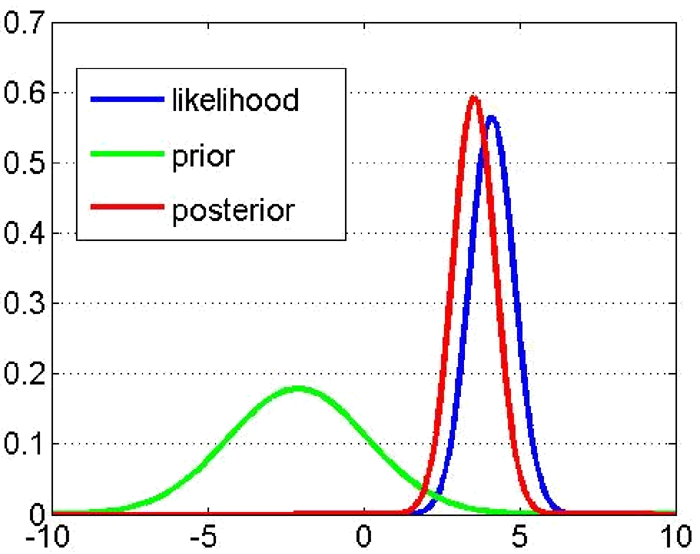


generative model M



Likelihood $Y = X\theta + \varepsilon$ $\quad \varepsilon \sim N(0,\gamma)$

Prior $\theta \sim N(\mu,\sigma)$



Qualifying priors

$$P(\theta|Y,M) = \frac{P(Y|\theta,M)P(\theta|M)}{P(Y|M)}$$

Shrinkage prior $\theta \sim N(0,\sigma)$

Uninformative (objective) prior $\theta \sim N(0,\sigma)$ with large σ

Conjugate prior when the prior and posterior distributions belong to the same family

Likelihood dist.	Conjugate prior dist.
Binomiale	Beta
Multinomiale	Dirichlet
Gaussian	Gaussian
Gamma	Gamma

Hierarchical models and empirical priors

Likelihood $Y = X\theta_1 + \varepsilon \qquad \varepsilon \sim N(0, \gamma)$

Prior $\theta = \{\theta_1, \theta_2, .., \theta_{k-1}\}$

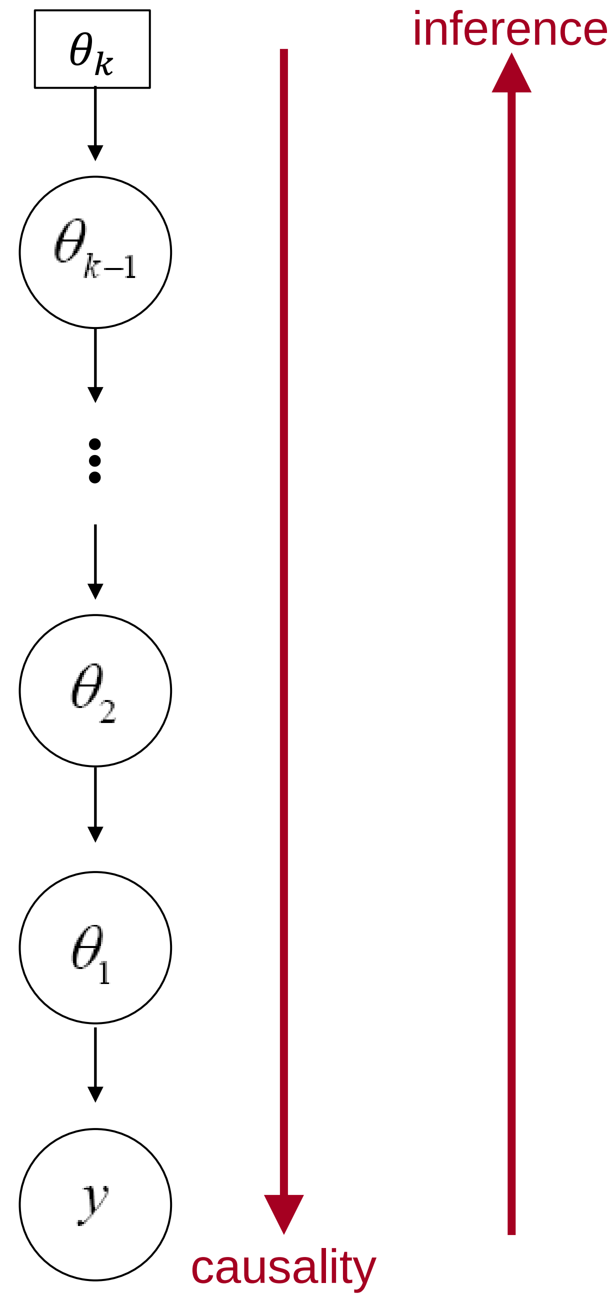
$$\theta_1 \sim N(\theta_2, \sigma_2)$$

$$\theta_2 \sim N(\theta_3, \sigma_3)$$

$$\vdots$$

$$\theta_{k-1} \sim N(\theta_k, \sigma_k)$$

Graphical representation



Another look at Bayes rule

Model/Hypothesis

Likelihood

Prior

$$P\left(\theta \middle| Y, M\right) = \frac{P\left(Y \middle| \theta, M\right) P\left(\theta \middle| M\right)}{P\left(Y \middle| M\right)}$$

Posterior
or conditional

Marginal likelihood
or evidence

To be infered

Model evidence and model posterior

$$P(\theta|Y,M)=\frac{P(Y|\theta,M)P(\theta|M)}{\boxed{P(Y|M)}}$$

Bayes rule again…

$$P(M|Y)=\frac{P(Y|M)P(M)}{P(Y)}$$

And with no prior in favor of one particular model…

$$P(M|Y)\propto P(Y|M)$$

Model comparison

if $P(Y|M_1) > P(Y|M_2)$, select model M_1

In practice, compute the Bayes Factor…

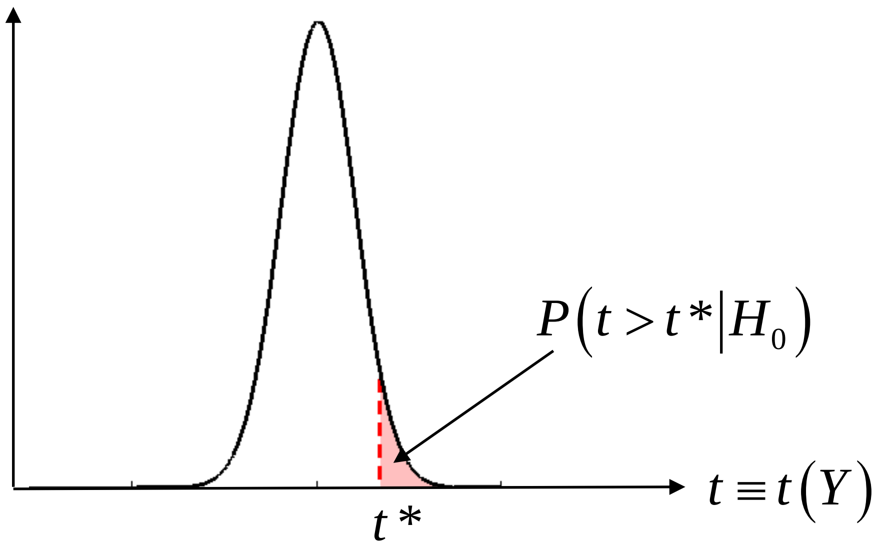
$$BF_{12} = \frac{P(Y|M_1)}{P(Y|M_2)}$$

… and apply the decision rule

B_{12}	Evidence
1 to 3	Weak
3 to 20	Positive
20 to 150	Strong
$\geq$ 150	Very strong

Hypothesis testing (classical way)

• given a null hypothesis, e.g.: $H_0 : \theta = 0$



• apply decision rule, i.e.:

if $P(t > t^* | H_0) \le \alpha$ then reject H0

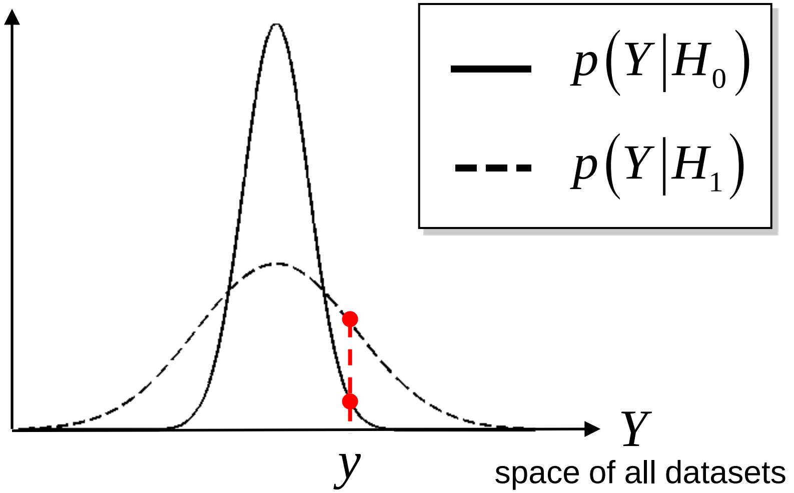
Statistical Parametric Map (SPM)

Hypothesis testing (bayesian way)

- define the null and the alternative hypothesis *in terms of priors*, e.g.:

$$H_0 : p(\theta|H_0) = \begin{cases} 1 & \text{if } \theta = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H_1 : p(\theta|H_1) = N(0, \Sigma)$$



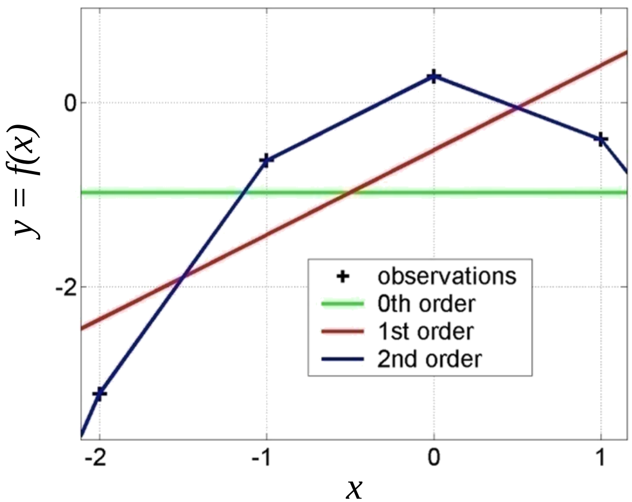
- apply decision rule, i.e.: if $\frac{P(y|H_0)}{P(y|H_1)} < u$ then reject H0

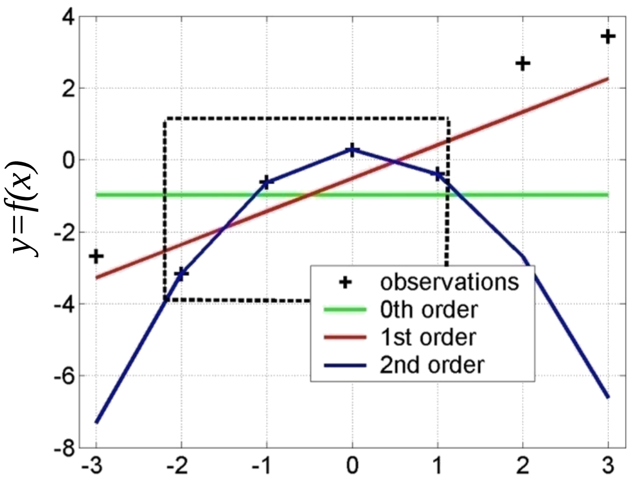
Principle of parsimony

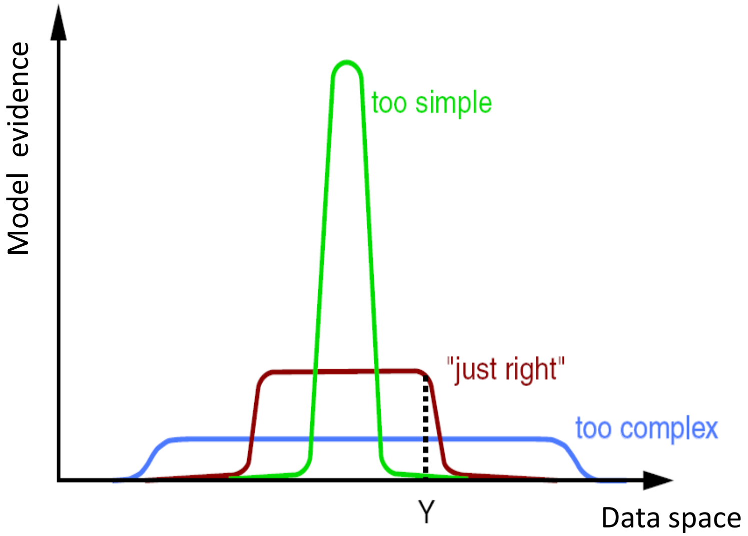
$$P(\theta|Y,M) = \frac{P(Y|\theta,M)P(\theta|M)}{P(Y|M)}$$

Occam's razor

Complex models should not be considered without necessity







$$p(Y|M) = \int p(Y|\theta,M)\,p(\theta|M)\,d\theta$$

→ **Usually no exact analytic solution !!**

Approximations to the (log-)evidence

$$\Delta BIC = -2\log\left[\frac{\sup P(Y|\theta, M_1)}{\sup P(Y|\theta, M_2)}\right] - (n2 - n1)\log N$$

$$\Delta AIC = -2\log\left[\frac{\sup P(Y|\theta, M_1)}{\sup P(Y|\theta, M_2)}\right] - 2(n2 - n1)$$

Free energy **F** ⟵ Obtained from the Variational Bayes inference

Variational Bayes Inference

Variational Bayes (VB) ≡ Expectation Maximization (EM) ≡ Restricted Maximum Likelihood (ReML)

Main features

- Iterative optimization procedure
- Yields a twofold inference on parameters θ and models M
- Uses a fixed-form approximate posterior $q(\theta)$
- Make use of approximations (e.g. mean field, Laplace)

to approach $P(\theta|Y,M)$ and $P(Y|M)$

The criterion to be maximized is the free-energy F

F is a lower bound to the log-evidence

$$\boldsymbol{F} = \ln P(Y|M) - D_{KL}\big(Q(\theta); P(\theta|Y,M)\big)$$
$$= \langle \ln P(Y|\theta, M)\rangle_Q - D_{KL}\big(Q(\theta); P(\theta|M)\big)$$

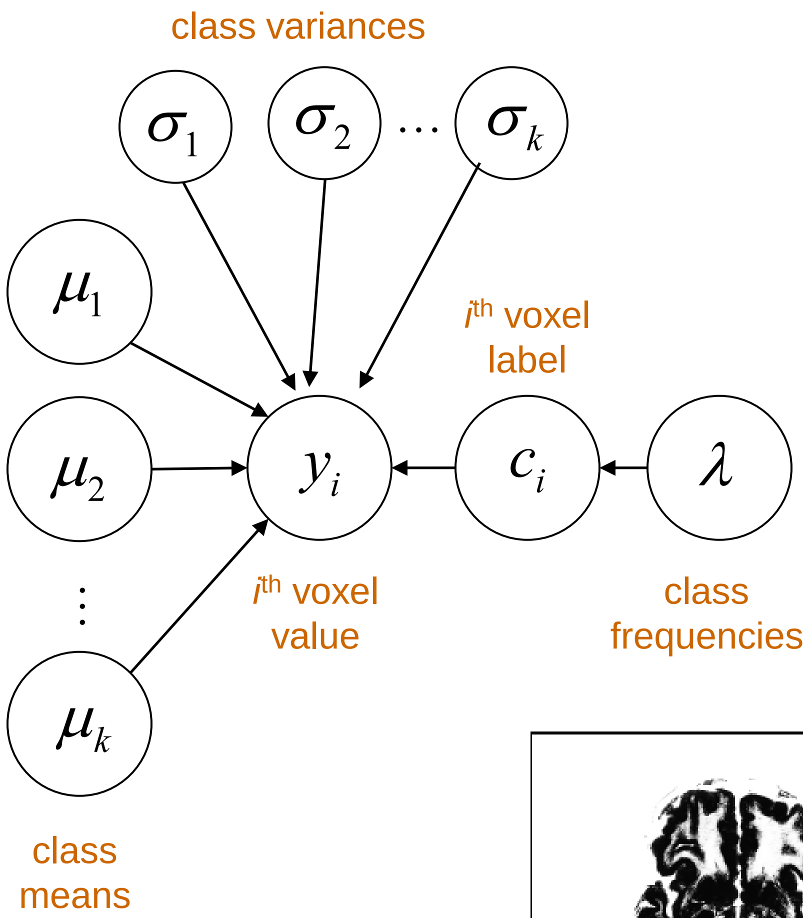
F = accuracy - complexity

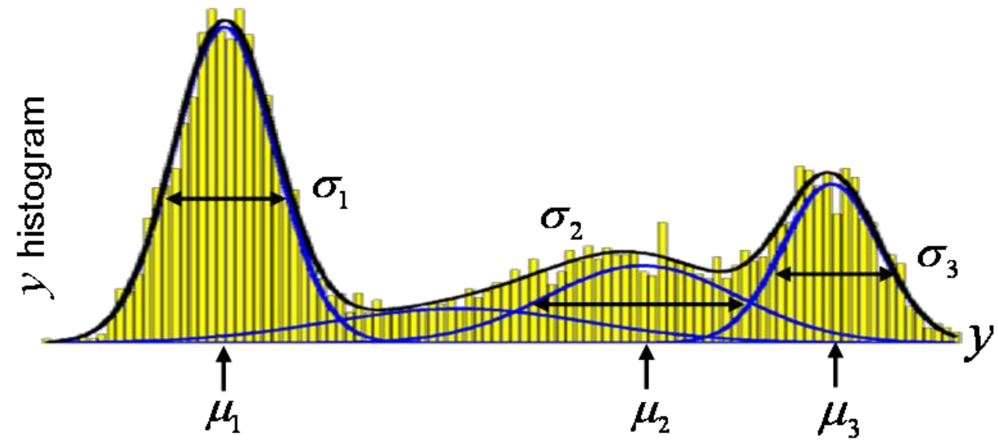
- General principles

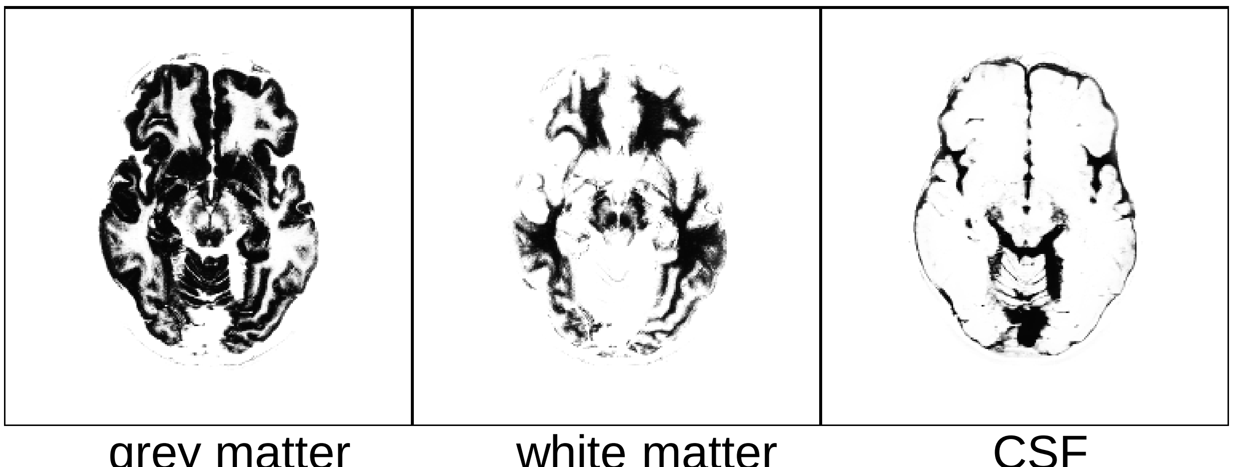
- The Bayesian way

- **SPM examples**

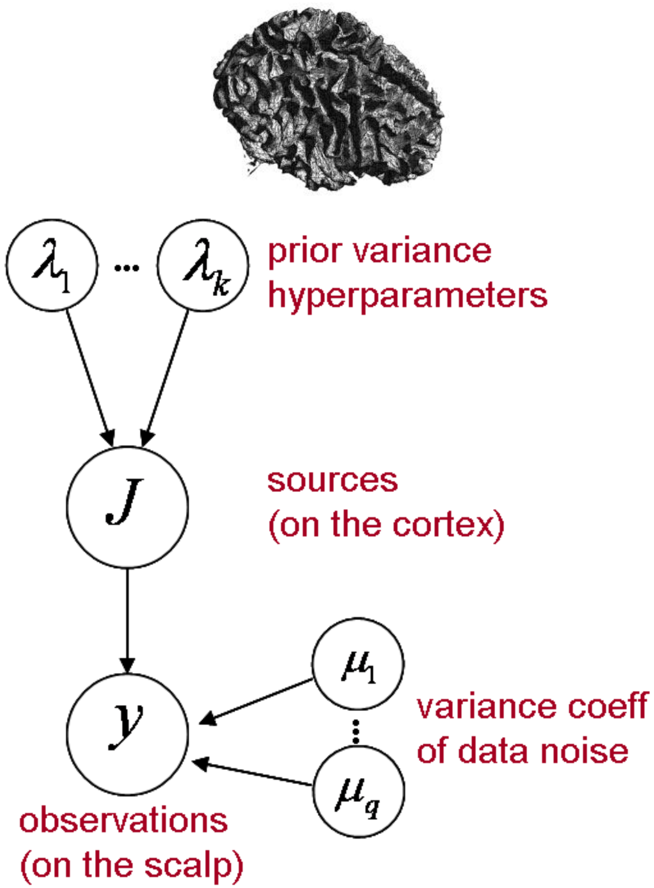
Segmentation of anatomical MRI

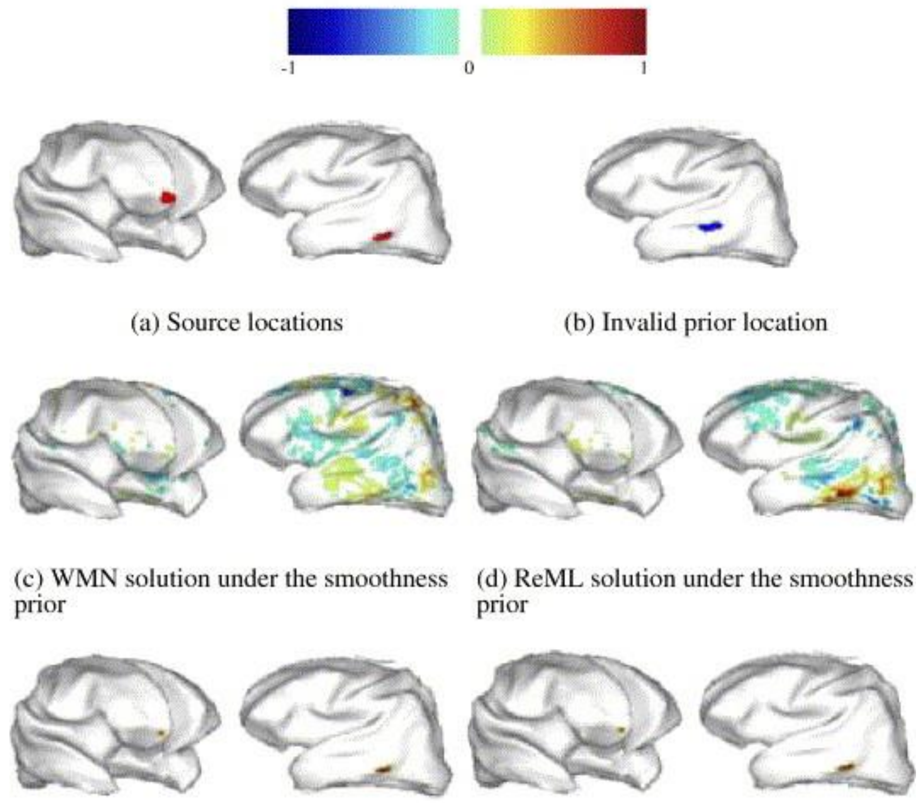






EEG/MEG source reconstruction

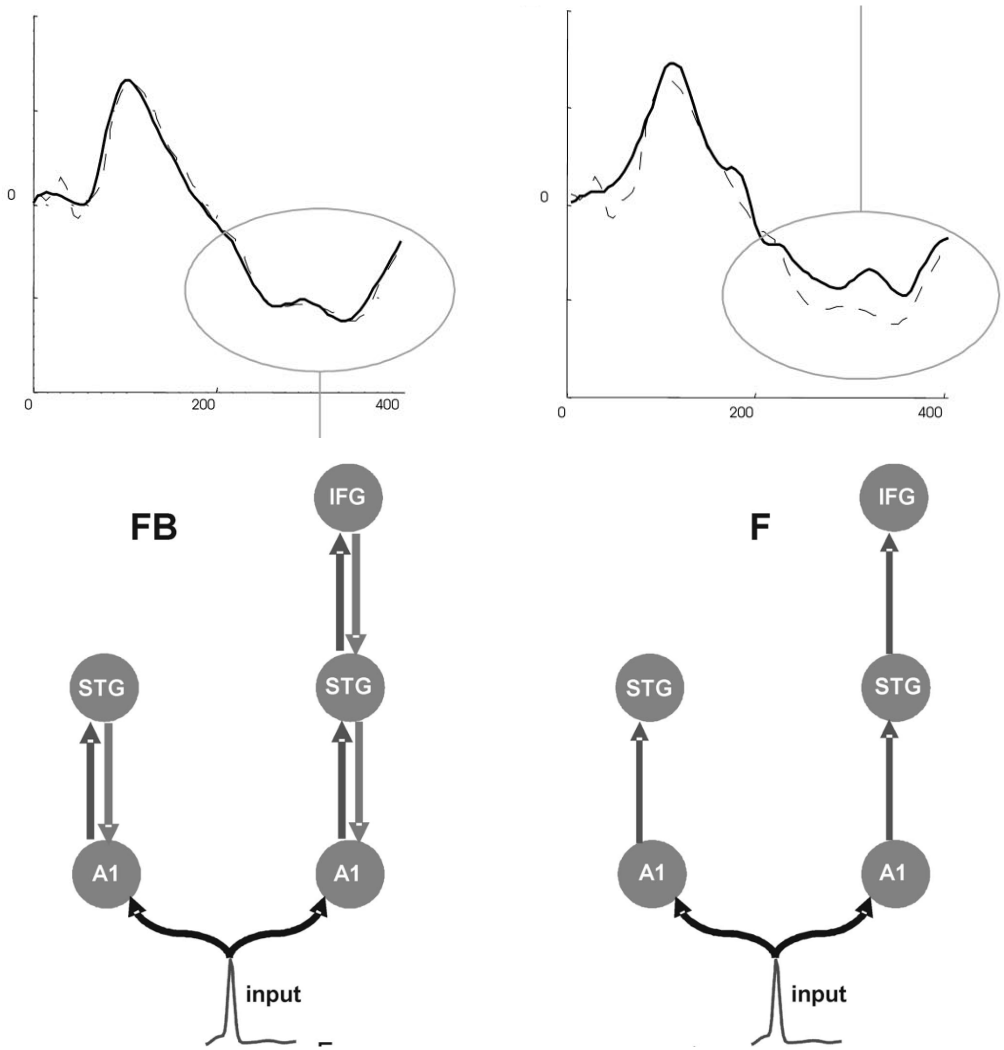




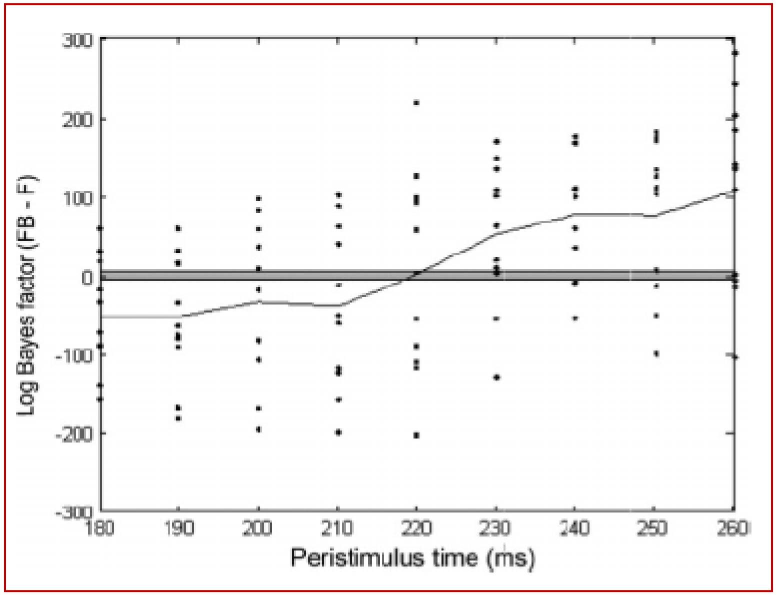
(a) Source locations
(b) Invalid prior location
(c) WMN solution under the smoothness prior
(d) ReML solution under the smoothness prior
(e) ReML solution under the smoothness and valid priors
(f) ReML solution under the smoothness, valid and invalid priors

Dynamic causal modelling of EEG data

Evidence for feedback loops (MMN paradigm)



Devient condition



Suggestions for further reading