

Bayesian Model Selection and Averaging

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SPM short course for M/EEG, London 2014

Model Comparison

Model Evidence

Complexity

Nonlinear Models

Bayes factors

Example

Families

FFX Model
Inference

RFX Model
Inference

Example

PXPs

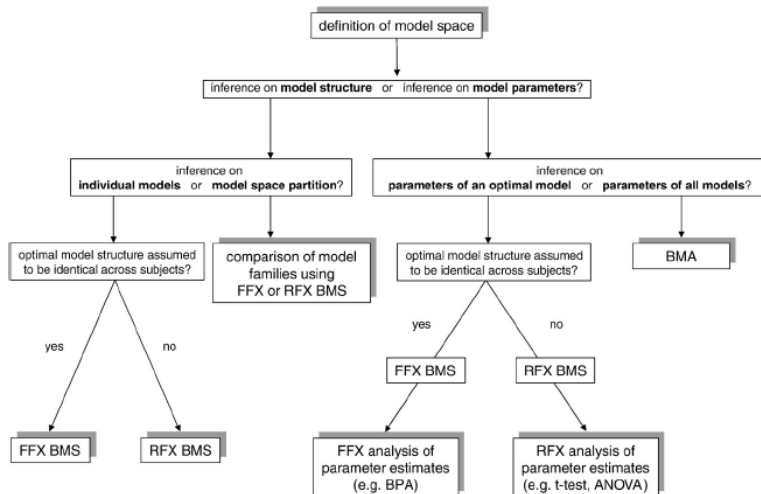
Model Averaging

RFX Parameter
Inference

FFX Parameter
Inference

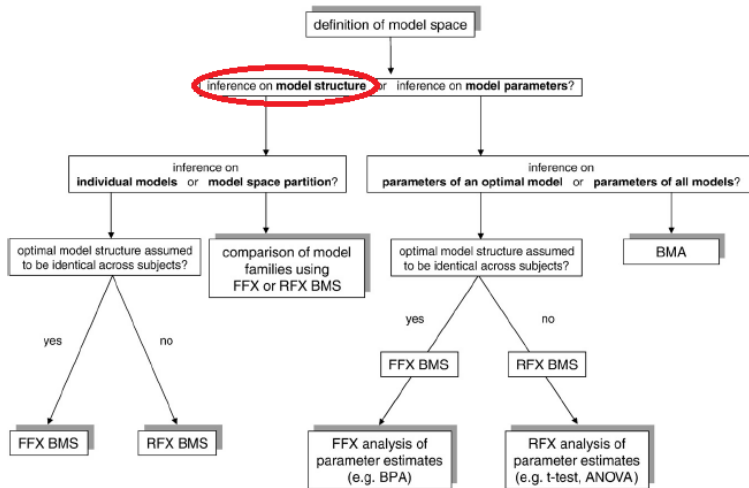
References

Ten Simple Rules



Stephan et al. Neuroimage, 2010

Model Structure



Model Comparison

Model Evidence
Complexity
Nonlinear Models
Bayes factors
Example

Families

FFX Model Inference

RFX Model Inference

Example
PXPs

Model Averaging

RFX Parameter Inference

FFX Parameter Inference

References

The model evidence is given by integrating out the dependence on model parameters

$$\begin{aligned} p(y|m) &= \int p(y, \theta|m) d\theta \\ &= \int p(y|\theta, m) p(\theta|m) d\theta \end{aligned}$$

Because we have marginalised over θ the evidence is also known as the marginal likelihood.

For linear Gaussian models there is an analytic expression for the model evidence.

Linear Models

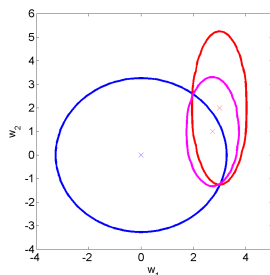
For Linear Models

$$y = Xw + e$$

where X is a design matrix and w are now regression coefficients. For prior mean μ_w , prior covariance C_w , observation noise covariance C_y the posterior distribution is given by

$$S_w^{-1} = X^T C_y^{-1} X + C_w^{-1}$$

$$m_w = S_w \left(X^T C_y^{-1} y + C_w^{-1} \mu_w \right)$$



The log model evidence comprises sum squared precision weighted prediction errors and Occam factors

$$\begin{aligned}\log p(y|m) &= -\frac{1}{2}e_y^T C_y^{-1} e_y - \frac{1}{2} \log |C_y| - \frac{N_y}{2} \log 2\pi \\ &- \frac{1}{2}e_w^T C_w^{-1} e_w - \frac{1}{2} \log \frac{|C_w|}{|S_w|}\end{aligned}$$

where prediction errors are the difference between what is expected and what is observed

$$e_y = y - X m_w$$

$$e_w = m_w - \mu_w$$

Bishop, Pattern Recognition and Machine Learning, 2006

Accuracy and Complexity

The log evidence for model m can be split into an accuracy and a complexity term

$$\log p(y|m) = \textit{Accuracy}(m) - \textit{Complexity}(m)$$

where

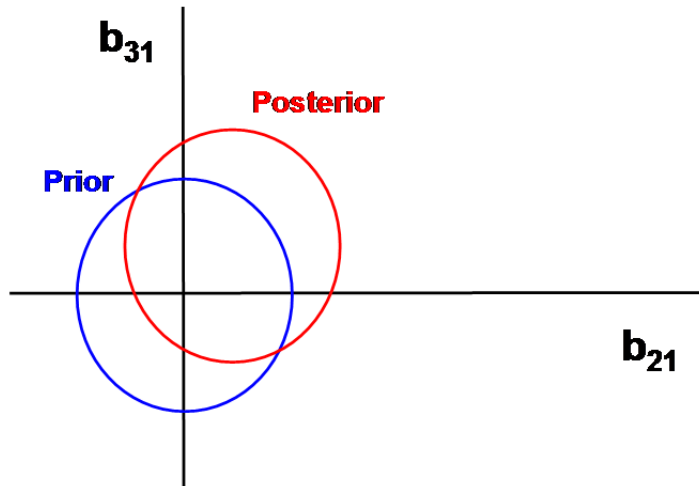
$$\textit{Accuracy}(m) = -\frac{1}{2} \mathbf{e}_y^T \mathbf{C}_y^{-1} \mathbf{e}_y - \frac{1}{2} \log |\mathbf{C}_y| - \frac{N_y}{2} \log 2\pi$$

and

$$\begin{aligned} \textit{Complexity}(m) &= \frac{1}{2} \mathbf{e}_w^T \mathbf{C}_w^{-1} \mathbf{e}_w + \frac{1}{2} \log \frac{|\mathbf{C}_w|}{|\mathbf{S}_w|} \\ &\approx \textit{KL}(\textit{prior} || \textit{posterior}) \end{aligned}$$

The Kullback-Leibler divergence measures the distance between probability distributions.

Small KL



Model Comparison

Model Evidence

Complexity

Nonlinear Models

Bayes factors

Example

Families

FFX Model

Inference

RFX Model

Inference

Example

PXPs

Model Averaging

RFX Parameter

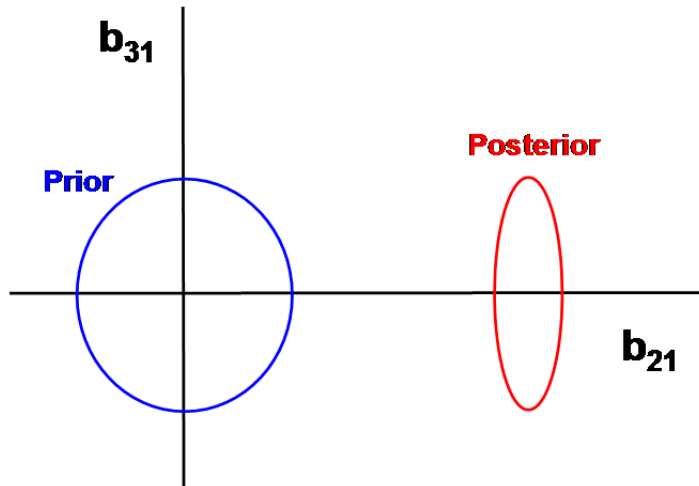
Inference

FFX Parameter

Inference

References

Medium KL



Model Comparison

Model Evidence

Complexity

Nonlinear Models

Bayes factors

Example

Families

FFX Model
Inference

RFX Model
Inference

Example

PXPs

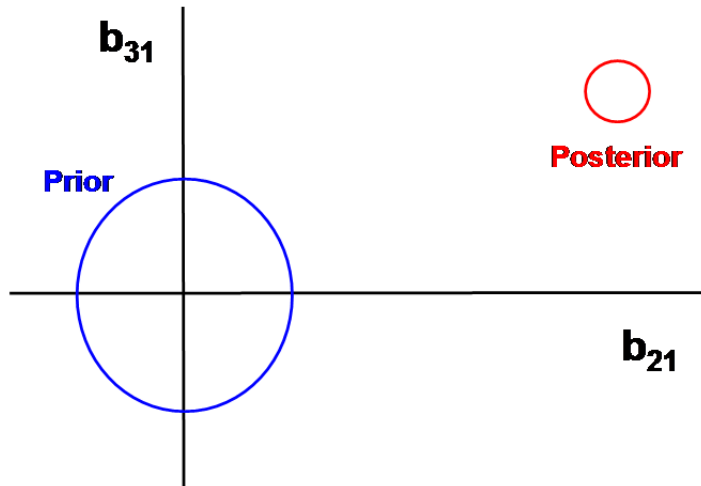
Model Averaging

RFX Parameter
Inference

FFX Parameter
Inference

References

Big KL



Nonlinear Models

For nonlinear models, we replace the true posterior with the approximate posterior (m_w, S_w) , and the previous expression becomes an approximation to the log model evidence called the (negative) Free Energy

$$F = -\frac{1}{2} e_y^T C_y^{-1} e_y - \frac{1}{2} \log |C_y| - \frac{N_y}{2} \log 2\pi \\ - \frac{1}{2} e_w^T C_w^{-1} e_w - \frac{1}{2} \log \frac{|C_w|}{|S_w|}$$

where

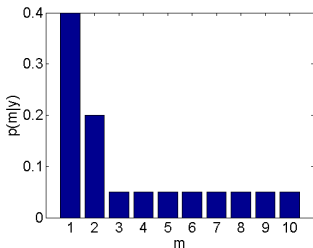
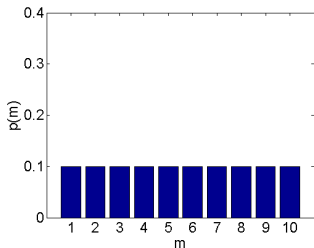
$$e_y = y - g(m_w)$$

$$e_w = m_w - \mu_w$$

and $g(m_w)$ is the DCM prediction. This is used to approximate the model evidence for DCMs. *Penny, Neuroimage, 2011.*

Bayes rule for models

A prior distribution over model space $p(m)$ (or ‘hypothesis space’) can be updated to a posterior distribution after observing data y .



This is implemented using Bayes rule

$$p(m|y) = \frac{p(y|m)p(m)}{p(y)}$$

where $p(y|m)$ is referred to as the evidence for model m and the denominator is given by

$$p(y) = \sum_{m'} p(y|m')p(m')$$

Bayes Factors

The Bayes factor for model j versus i is the ratio of model evidences

$$B_{ji} = \frac{p(y|m=j)}{p(y|m=i)}$$

We have

$$B_{ij} = \frac{1}{B_{ji}}$$

Hence

$$\begin{aligned} \log B_{ji} &= \log p(y|m=j) - \log p(y|m=i) \\ &= F_j - F_i \end{aligned}$$

Model Comparison

Model Evidence

Complexity

Nonlinear Models

Bayes factors

Example

Families

FFX Model Inference

RFX Model Inference

Example

PXPs

Model Averaging

RFX Parameter Inference

FFX Parameter Inference

References

Posterior Model Probability

Given equal priors, $p(m = i) = p(m = j)$ the posterior model probability is

$$\begin{aligned} p(m = i|y) &= \frac{p(y|m = i)}{p(y|m = i) + p(y|m = j)} \\ &= \frac{1}{1 + \frac{p(y|m=j)}{p(y|m=i)}} \\ &= \frac{1}{1 + B_{ji}} \\ &= \frac{1}{1 + \exp(\log B_{ji})} \\ &= \frac{1}{1 + \exp(-\log B_{ij})} \end{aligned}$$

Model Comparison

Model Evidence

Complexity

Nonlinear Models

Bayes factors

Example

Families

FFX Model
Inference

RFX Model
Inference

Example

PXPs

Model Averaging

RFX Parameter
Inference

FFX Parameter
Inference

References

Posterior Model Probability

Hence

$$p(m = i|y) = \sigma(\log B_{ij})$$

where B_{ij} is the Bayes factor for model i versus model j and

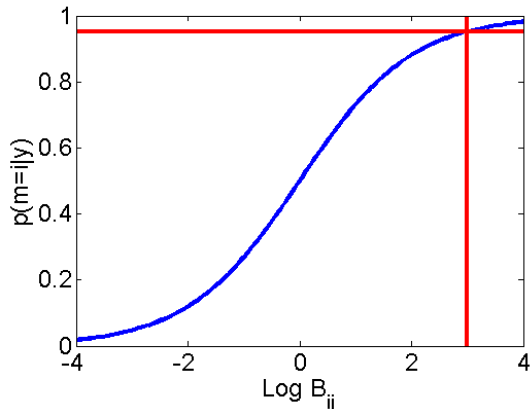
$$\sigma(x) = \frac{1}{1 + \exp(-x)}$$

is the sigmoid function.

Bayes factors

The posterior model probability is a sigmoidal function of the log Bayes factor

$$p(m = i | y) = \sigma(\log B_{ij})$$



Bayes factors

The posterior model probability is a sigmoidal function of the log Bayes factor

$$p(m = i|y) = \sigma(\log B_{ij})$$

Table 1
Interpretation of Bayes factors

B_{ij}	$p(m = i y)$ (%)	Evidence in favor of model i
1–3	50–75	Weak
3–20	75–95	Positive
20–150	95–99	Strong
≥ 150	≥ 99	Very strong

Bayes factors can be interpreted as follows. Given candidate hypotheses i and j , a Bayes factor of 20 corresponds to a belief of 95% in the statement ‘hypothesis i is true’. This corresponds to strong evidence in favor of i .

Kass and Raftery, JASA, 1995.

Odds Ratios

If we don't have uniform priors one can work with odds ratios.

The prior and posterior odds ratios are defined as

$$\pi_{ij}^0 = \frac{p(m=i)}{p(m=j)}$$
$$\pi_{ij} = \frac{p(m=i|y)}{p(m=j|y)}$$

respectively, and are related by the Bayes Factor

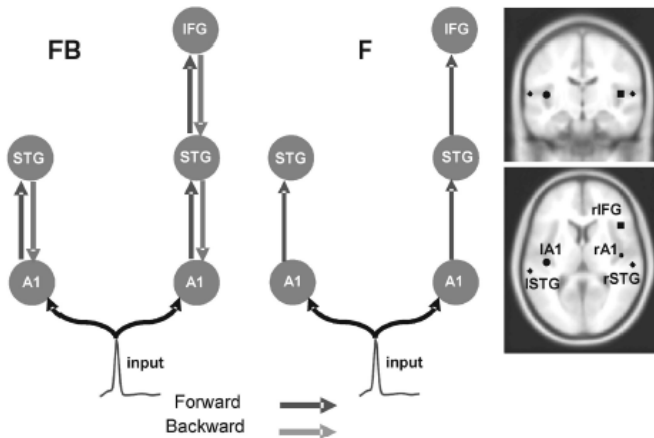
$$\pi_{ij} = B_{ij} \times \pi_{ij}^0$$

eg. priors odds of 2 and Bayes factor of 10 leads posterior odds of 20.

An odds ratio of 20 is 20-1 ON in bookmakers parlance.

Example

Modelling auditory responses with DCM for ERP



Garrido et al, PNAS, 2007

Model Comparison

Model Evidence
Complexity
Nonlinear Models
Bayes factors
Example

Families

FFX Model
Inference

RFX Model
Inference

Example
PXPs

Model Averaging

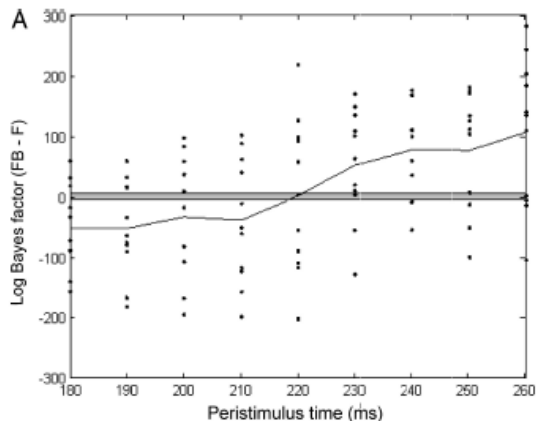
RFX Parameter
Inference

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References

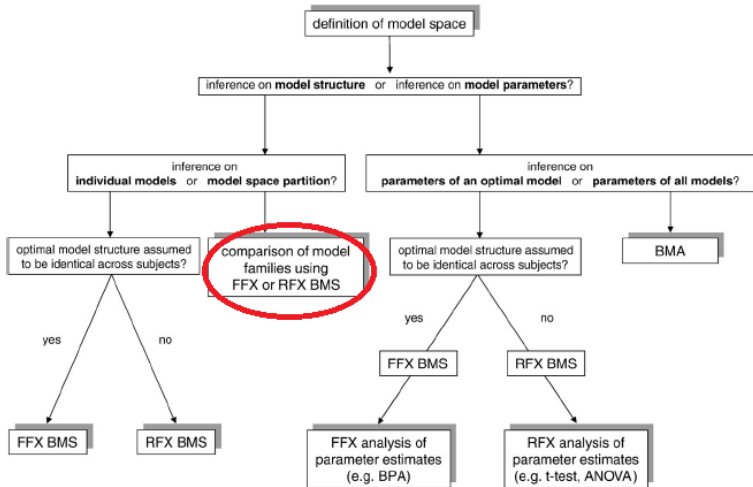
Example

Train DCMs from stimulus onset up to peristimulus time T . FB model favoured more heavily as T increases.



Evoked responses are generated by feedback loops.

Families



Model Comparison

Model Evidence
Complexity
Nonlinear Models
Bayes factors
Example

Families

FFX Model Inference

RFX Model Inference

Example
PXPs

Model Averaging

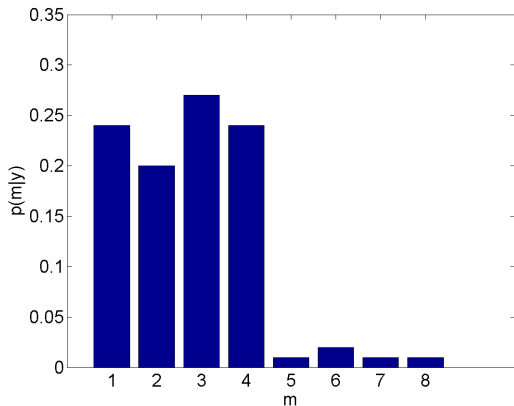
RFX Parameter Inference

FFX Parameter Inference

References

Posterior Model Probabilities

Say we've fitted 8 DCMs and get the following distribution over models

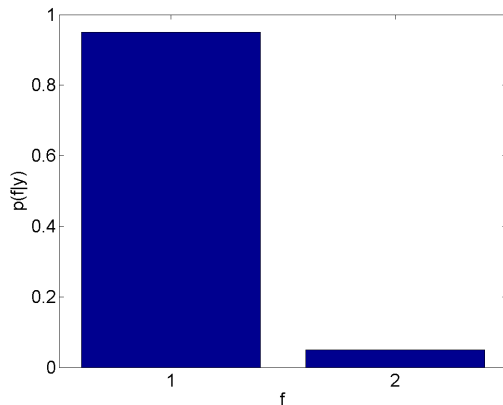


Similar models share probability mass (dilution). The probability for any single model can become very small esp. for large model spaces.

Model Families

Assign model m to family f eg. first four to family one, second four to family two. The posterior family probability is then

$$p(f|y) = \sum_{m \in S_f} p(m|y)$$



Different Sized Families

If we have K families, then to avoid bias in family inference we wish to have a uniform prior at the family level

$$p(f) = \frac{1}{K}$$

The prior family probability is related to the prior model probability

$$p(f) = \sum_{m \in \mathcal{S}_f} p(m)$$

where the sum is over all N_f models in family f . So we set

$$p(m) = \frac{1}{KN_f}$$

for all models in family f before computing $p(m|y)$. This allows us to have families with unequal numbers of models. *Penny et al. PLOS-CB, 2010.*

Different Sized Families

So say we have two families. We want a prior for each family of $p(f) = 0.5$.

If family one has $N_1 = 2$ models and family two has $N_2 = 8$ models, then we set

$$p(m) = \frac{1}{2} \times \frac{1}{2} = 0.25$$

for all models in family one and

$$p(m) = \frac{1}{2} \times \frac{1}{8} = 0.0625$$

for all models in family two.

These are then used in Bayes rule for models

$$p(m|y) = \frac{p(y|m)p(m)}{p(y)}$$

Model Comparison

- Model Evidence
- Complexity
- Nonlinear Models
- Bayes factors
- Example

Families

FFX Model Inference

RFX Model Inference

- Example
- PXPs

Model Averaging

RFX Parameter Inference

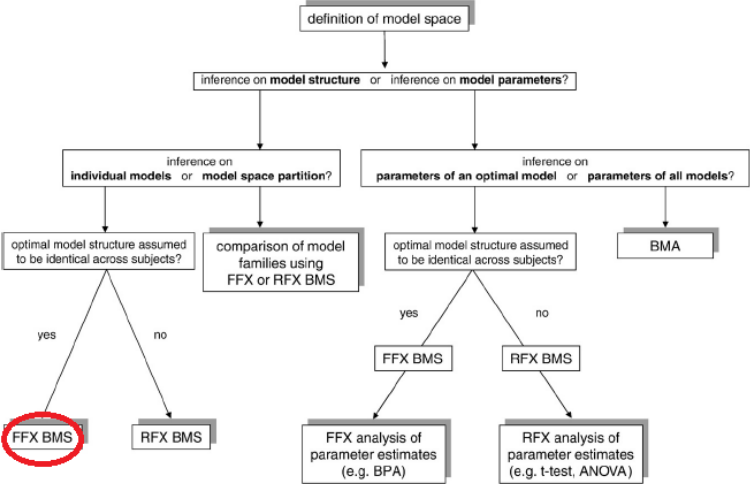
FFX Parameter Inference

References

Fixed Effects BMS

- Model Evidence
- Complexity
- Nonlinear Models
- Bayes factors
- Example

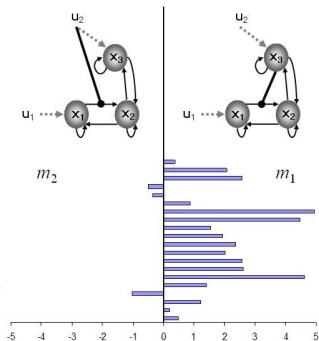
- Example
- PXPs



Fixed Effects BMS

Two models, twenty subjects.

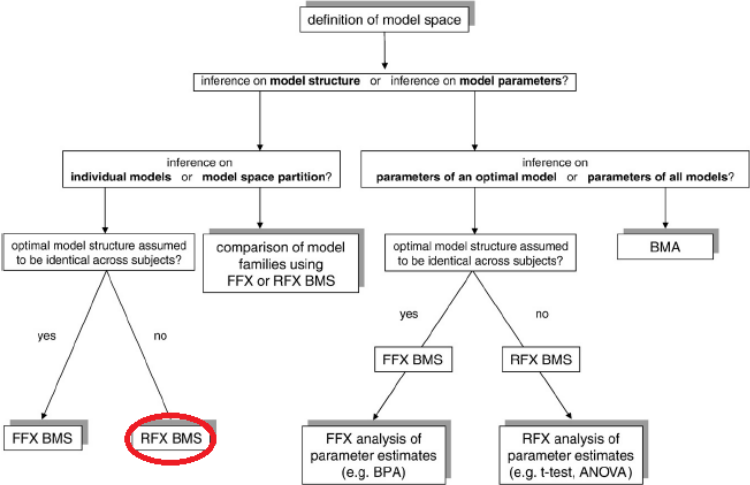
$$\log p(Y|m) = \sum_{n=1}^N \log p(y_n|m)$$



The Group Bayes Factor (GBF) is

$$B_{ij} = \prod_{n=1}^N B_{ij}(n)$$

Random Effects BMS



Model Comparison

- Model Evidence
- Complexity
- Nonlinear Models
- Bayes factors
- Example

Families

FFX Model Inference

RFX Model Inference

- Example
- PXPs

Model Averaging

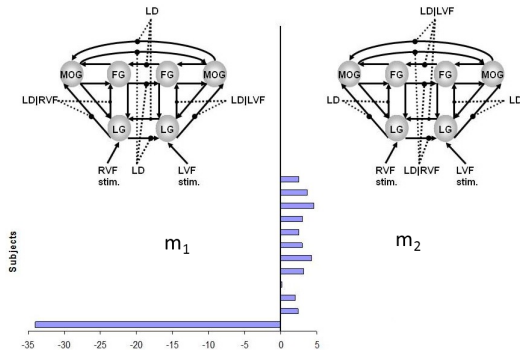
RFX Parameter Inference

FFX Parameter Inference

References

Random Effects BMS

Stephan et al. *J. Neurosci*, 2007



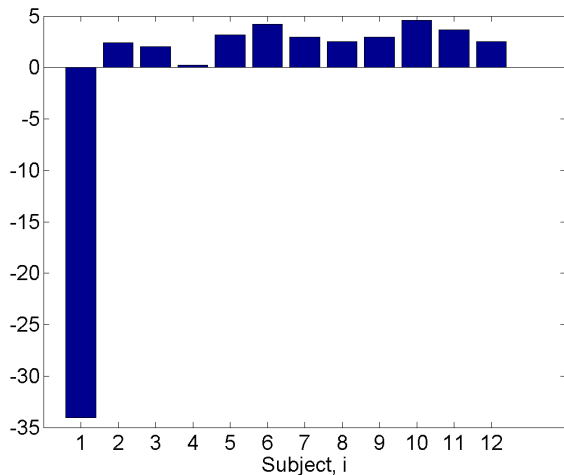
11/12=92% subjects favour model 2.

$GBF = 15$ in favour of model 1. FFX inference does not agree with the majority of subjects.

RFX Model Inference

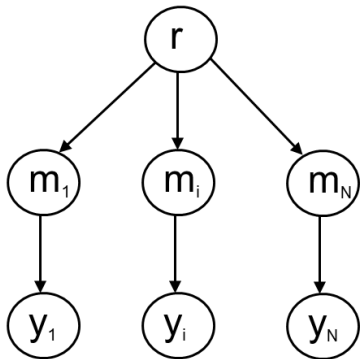
Log Bayes Factor in favour of model 2

$$\log \frac{p(y_i | m_i = 2)}{p(y_i | m_i = 1)}$$



RFX Model Inference

Model frequencies r_k , model assignments m_i , subject data y_i .



Approximate posterior

$$q(r, m|Y) = q(r|Y)q(m|Y)$$

Stephan et al, Neuroimage, 2009.

RFX Model Inference

Model Comparison

- Model Evidence
- Complexity
- Nonlinear Models
- Bayes factors
- Example

Families

FFX Model Inference

RFX Model Inference

- Example
- PXPs

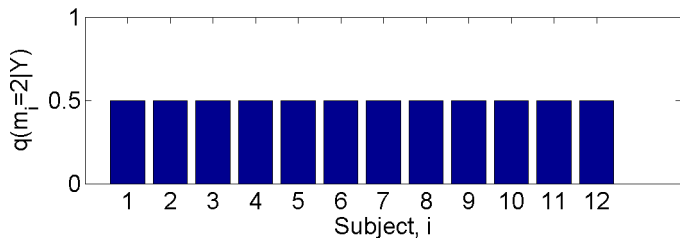
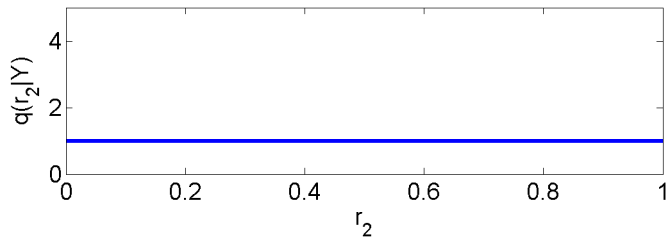
Model Averaging

RFX Parameter Inference

FFX Parameter Inference

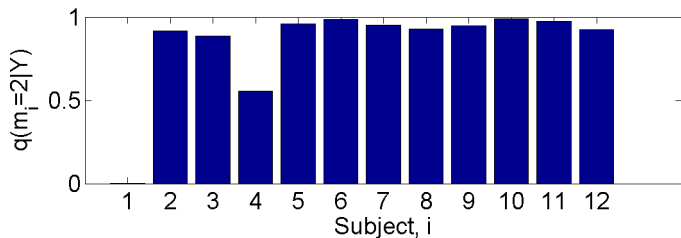
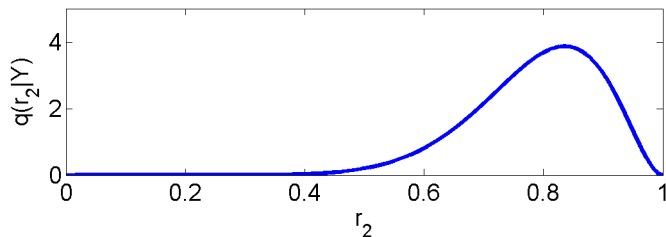
References

Iteration 0



RFX Model Inference

Iteration 1



Model Comparison

- Model Evidence
- Complexity
- Nonlinear Models
- Bayes factors
- Example

Families

- FFX Model Inference

RFX Model Inference

- Example
- PXPs

Model Averaging

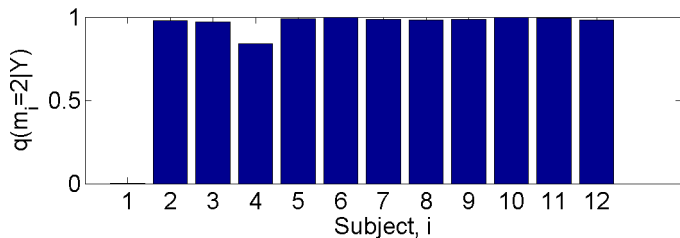
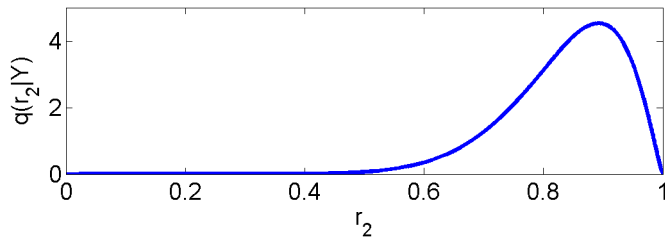
- RFX Parameter Inference

- FFX Parameter Inference

- References

RFX Model Inference

Iteration 2



Model Comparison

- Model Evidence
- Complexity
- Nonlinear Models
- Bayes factors
- Example

Families

- FFX Model Inference

RFX Model Inference

- Example
- PXPs

Model Averaging

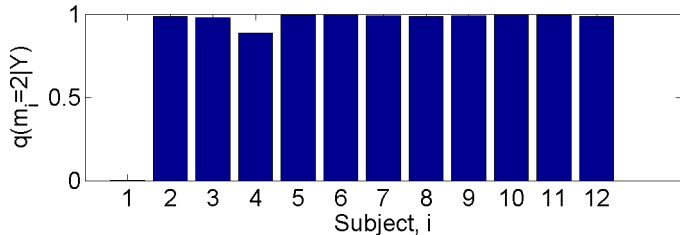
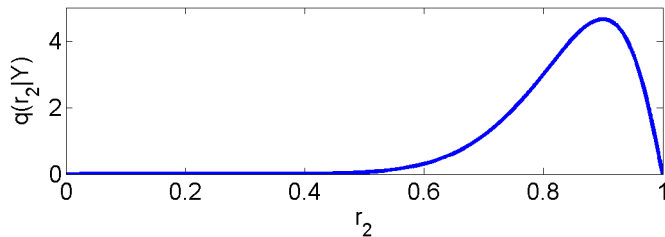
- RFX Parameter Inference

- FFX Parameter Inference

- References

RFX Model Inference

Iteration 3



Model Comparison

- Model Evidence
- Complexity
- Nonlinear Models
- Bayes factors
- Example

Families

- FFX Model Inference

RFX Model Inference

- Example
- PXPs

Model Averaging

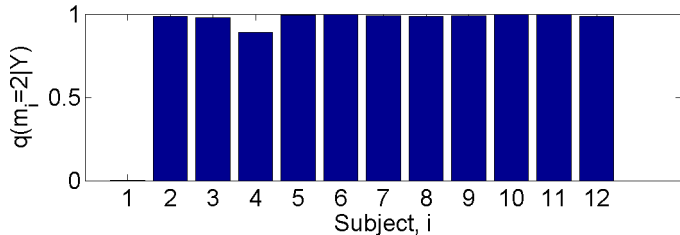
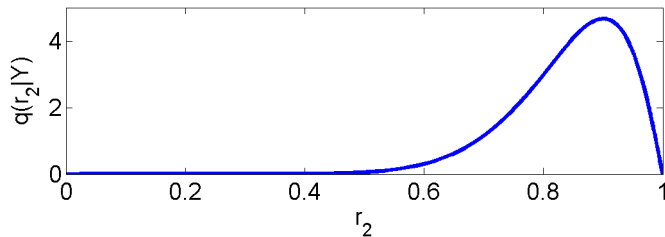
- RFX Parameter Inference

- FFX Parameter Inference

- References

RFX Model Inference

Iteration 4



Model Comparison

- Model Evidence
- Complexity
- Nonlinear Models
- Bayes factors
- Example

Families

- FFX Model Inference

RFX Model Inference

- Example
- PXPs

Model Averaging

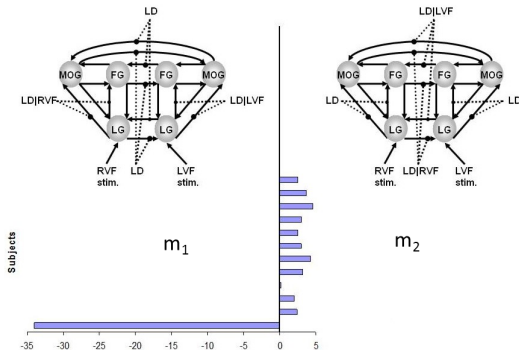
- RFX Parameter Inference

- FFX Parameter Inference

- References

Random Effects

11/12=92% subjects favoured model 2.



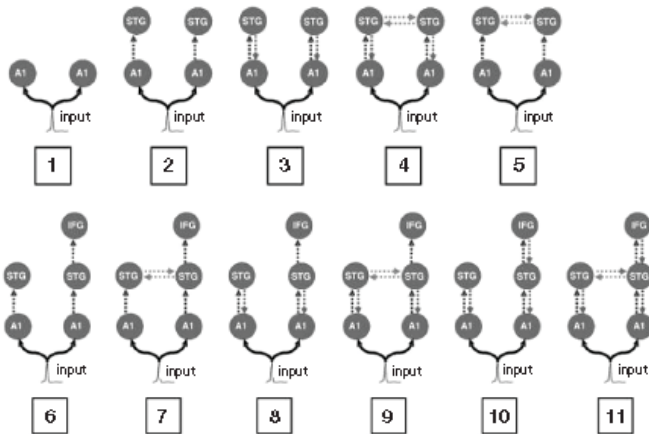
$$E[r_2 | Y] = 0.84$$

$$p(r_2 > r_1 | Y) = 0.99$$

where the latter is called the exceedance probability.

Example

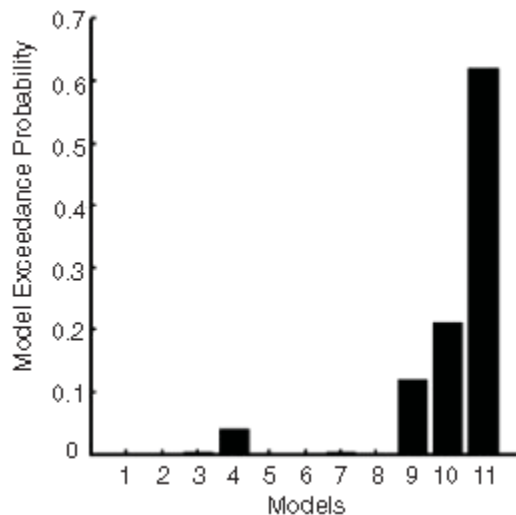
Auditory responses to stimuli with 'roving' frequencies modelled with DCM for ERP.



Boly et al, Science, 2011.

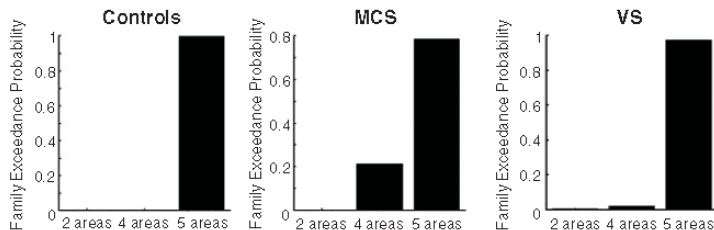
Example

Model Exceedance Probabilities



Example

Family inference - number of regions



This study used people in a Minimally Conscious State (MCS), in a Vegetative State (VS) or in a normal level of consciousness (Controls).

Model Comparison

- Model Evidence
- Complexity
- Nonlinear Models
- Bayes factors
- Example

Families

- FFX Model Inference

- RFX Model Inference

- Example
- PXPs

Model Averaging

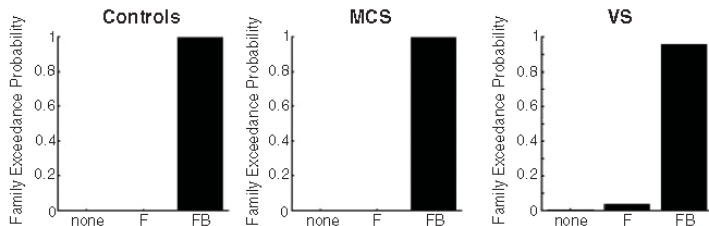
- RFX Parameter Inference

- FFX Parameter Inference

- References

Example

Family inference - type of connections



This study used people in a Minimally Conscious State (MCS), in a Vegetative State (VS) or in a normal level of consciousness (Controls).

Model Comparison

- Model Evidence
- Complexity
- Nonlinear Models
- Bayes factors
- Example

Families

- FFX Model Inference

- RFX Model Inference

- Example
- PXPs

Model Averaging

- RFX Parameter Inference

- FFX Parameter Inference

- References

Protected Exceedance Probabilities

The use of Exceedance Probabilities (xp's) assumes the frequencies are different for each model.

But what if the model frequencies are all the same ? (H_0 : omnibus hypothesis)

Let $p_0 = p(H_0|Y)$. Then the (posterior) probability that frequencies are different is $1 - p_0$.

Rigoux et al. (*Neuroimage, 2014*) show how to compute p_0 and then define Protected Exceedance Probabilities as

$$pxp = xp(1 - p_0) + \frac{1}{K}p_0$$

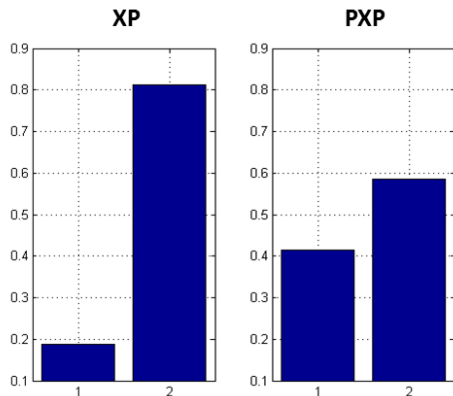
where K is the number of models.

p_0 also referred to as 'Bayes Omnibus Risk (BOR)'.

Protected Exceedance Probabilities

The function `spm_BMS.m` reports pxp 's and p_0 .

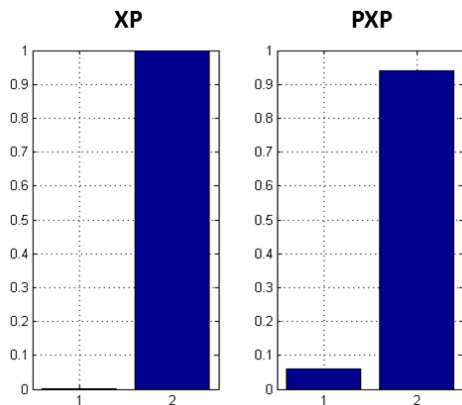
Synthetic data ($K = 2$ models, $N = 12$ subjects, mean log evidence difference=0).



We have $p_0 = 0.72$.

Protected Exceedance Probabilities

Synthetic data ($K = 2$ models, $N = 12$ subjects, mean log evidence difference=1).



We have $p_0 = 0.11$.

PXPs also very useful for large K .

Dependence on Comparison Set

The ranking of models from RFX inference can depend on the comparison set.

Say we have two models with 7 subjects preferring model 1 and 10 ten subjects preferring model 2. The model frequencies are $r_1 = 7/17 = 0.41$ and $r_2 = 10/17 = 0.59$.

Now say we add a third model which is similar to the second, and that 4 of the subjects that used to prefer model 2 now prefer model 3. The model frequencies are now $r_1 = 7/17 = 0.41$, $r_2 = 6/17 = 0.35$ and $r_3 = 4/17 = 0.24$.

This is like voting in elections.

Penny et al. PLOS-CB, 2010.

Model Comparison

Model Evidence

Complexity

Nonlinear Models

Bayes factors

Example

Families

FFX Model Inference

RFX Model Inference

Example

PXPs

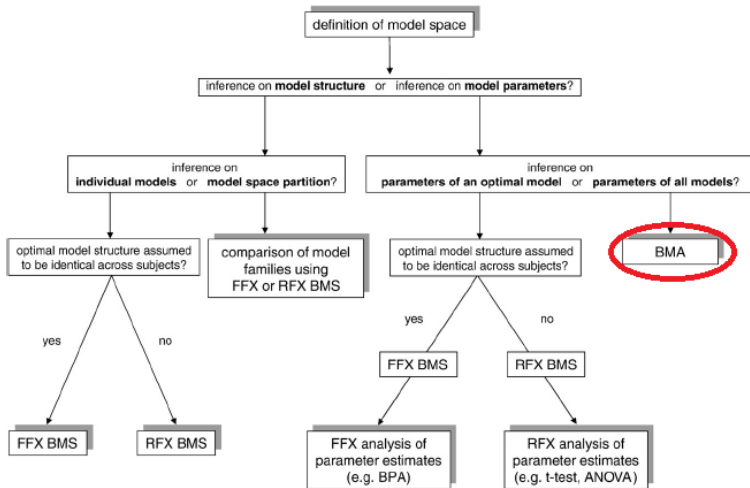
Model Averaging

RFX Parameter Inference

FFX Parameter Inference

References

Model Averaging



Model Comparison

Model Evidence
Complexity
Nonlinear Models
Bayes factors
Example

Families

FFX Model Inference

RFX Model Inference

Example
PXPs

Model Averaging

RFX Parameter Inference

FFX Parameter Inference

References

Model Averaging

Each DCM.mat file stores the posterior mean (DCM.Ep) and covariance (DCM.Cp) for each fitted model. This defines the posterior mean over parameters for that model, $p(\theta|m, y)$.

This can then be combined with the posterior model probabilities $p(m|y)$ to compute a posterior over parameters

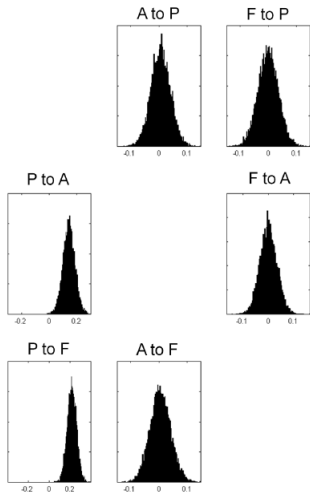
$$\begin{aligned} p(\theta|y) &= \sum_m p(\theta, m|y) \\ &= \sum_m p(\theta|m, y)p(m|y) \end{aligned}$$

which is independent of model assumptions (within the chosen set). Here, we marginalise over m .

The sum over m could be restricted to eg. models within the winning family.

Model Averaging

The distribution $p(\theta|y)$ can be gotten by sampling;
sample m from $p(m|y)$, then sample θ from $p(\theta|m, y)$.



If a connection doesn't exist for model m the relevant
samples are set to zero.

Model Comparison

- Model Evidence
- Complexity
- Nonlinear Models
- Bayes factors
- Example

Families

FFX Model Inference

RFX Model Inference

- Example
- PXPs

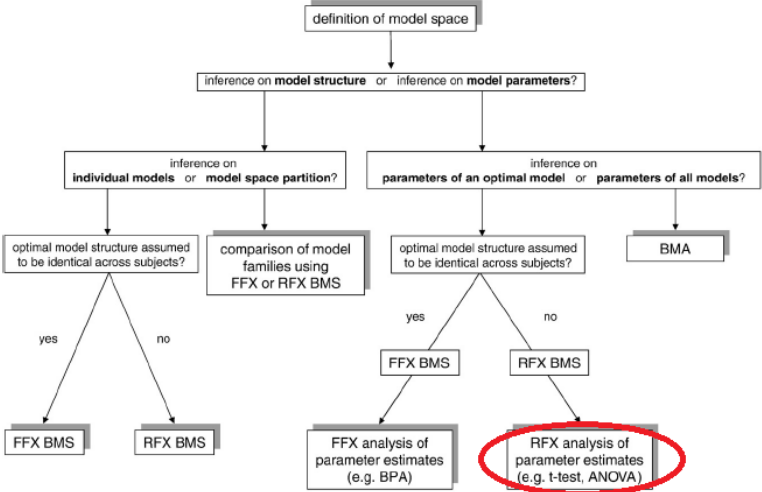
Model Averaging

RFX Parameter Inference

FFX Parameter Inference

References

RFX Parameter Inference



Model Comparison

- Model Evidence
- Complexity
- Nonlinear Models
- Bayes factors
- Example

Families

FFX Model Inference

RFX Model Inference

- Example
- PXPs

Model Averaging

RFX Parameter Inference

FFX Parameter Inference

References

RFX Parameter Inference

If i th subject has posterior mean value m_i we can use these in Summary Statistic approach for group parameter inference (eg two-sample t-tests for control versus patient inferences).

eg P to A connection in controls: 0.20, 0.12, 0.32, 0.11, 0.01, ...

eg P to A connection in patients: 0.50, 0.42, 0.22, 0.71, 0.31, ...

Two sample t-test shows the P to A connection is stronger in patients than controls ($p < 0.05$). Or one sample t-tests if we have a single group.

RFX is more conservative than BPA.

Model Comparison

Model Evidence
Complexity
Nonlinear Models
Bayes factors
Example

Families

FFX Model Inference

RFX Model Inference

Example
PXPs

Model Averaging

RFX Parameter Inference

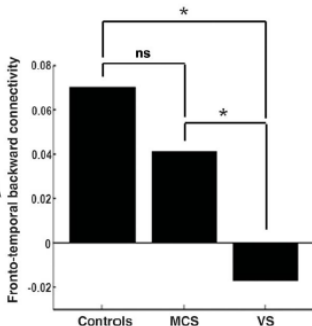
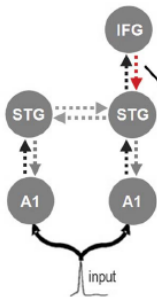
FFX Parameter Inference

References

Example

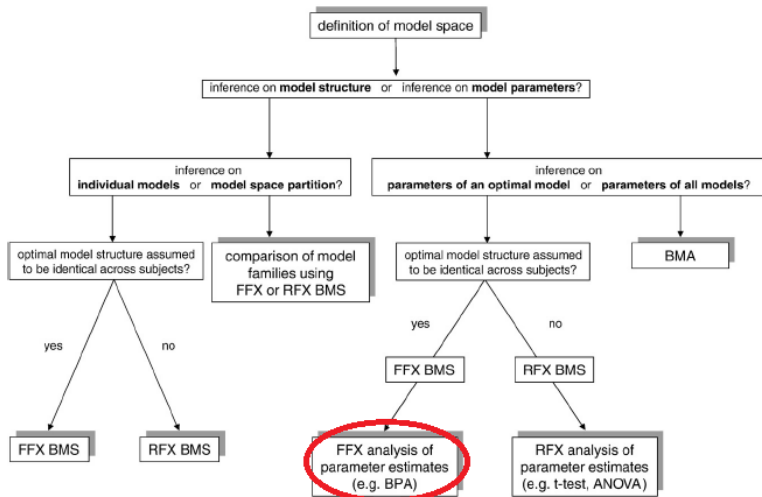
T-tests on backward connection from IFG to STG

Fig. 4. Quantitative effective connectivity analysis revealed that the only significant difference between VS patients and controls was an impairment of backward connectivity from frontal to temporal cortex. MCS subjects showed significantly preserved connectivity compared with VS subjects and were not significantly different from controls.



Boly et al. Science, 2011

FFX Parameter Inference



Model Comparison

Model Evidence
Complexity
Nonlinear Models
Bayes factors
Example

Families

FFX Model Inference

RFX Model Inference

Example
PXPs

Model Averaging

RFX Parameter Inference

FFX Parameter Inference

References

FFX Parameter Inference

RFX parameter inference (eg. t-tests, F-tests) - allow for variability over eg. subjects.

FFX parameter inference - assumes no variability over eg. subjects/sessions.

FFX parameter inference - implemented using 'Bayesian Parameter Averaging' (BPA)

Model Comparison

Model Evidence
Complexity
Nonlinear Models
Bayes factors
Example

Families

FFX Model Inference

RFX Model Inference

Example
PXPs

Model Averaging

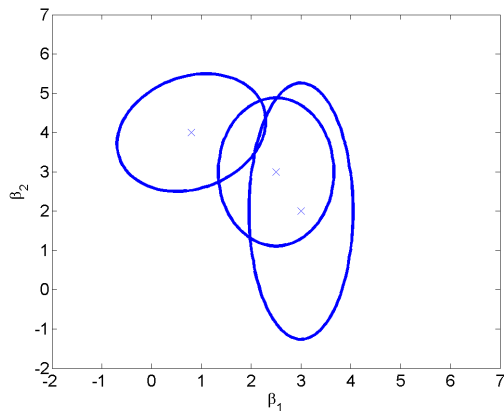
RFX Parameter Inference

FFX Parameter Inference

References

Bayesian Parameter Averaging

If for the i th subject the posterior mean and precision are μ_i and Λ_i



Three subjects shown.

Bayesian Parameter Averaging

If for the i th subject the posterior mean and precision are μ_i and Λ_i then the posterior mean and precision for the group are

$$\Lambda = \sum_{i=1}^N \Lambda_i$$
$$\mu = \Lambda^{-1} \sum_{i=1}^N \Lambda_i \mu_i$$

Kasses et al, Neuroimage, 2010.

This is a FFX analysis where each subject adds to the posterior precision.

Model Comparison

Model Evidence
Complexity
Nonlinear Models
Bayes factors
Example

Families

FFX Model Inference

RFX Model Inference

Example
PXPps

Model Averaging

RFX Parameter Inference

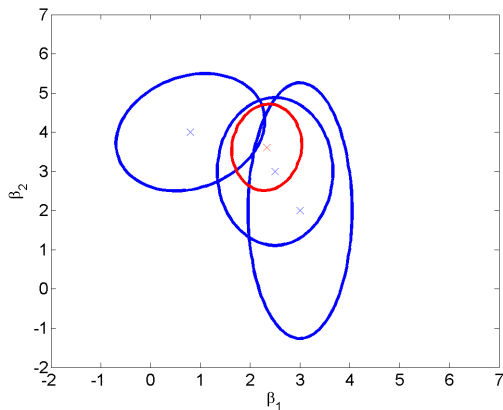
FFX Parameter Inference

References

Bayesian Parameter Averaging

$$\Lambda = \sum_{i=1}^N \Lambda_i$$

$$\mu = \Lambda^{-1} \sum_{i=1}^N \Lambda_i \mu_i$$



Model Comparison

- Model Evidence
- Complexity
- Nonlinear Models
- Bayes factors
- Example

Families

FFX Model Inference

RFX Model Inference

- Example
- PXPs

Model Averaging

RFX Parameter Inference

FFX Parameter Inference

References

If for the i th subject the posterior mean and precision are μ_i and Λ_i then the posterior mean and precision for the group are

$$\Lambda = \sum_{i=1}^N \Lambda_i - (N - 1)\Lambda_0$$
$$\mu = \Lambda^{-1} \left(\sum_{i=1}^N \Lambda_i \mu_i - (N - 1)\Lambda_0 \mu_0 \right)$$

Formulae augmented to accomodate non-zero priors Λ_0 and μ_0 .

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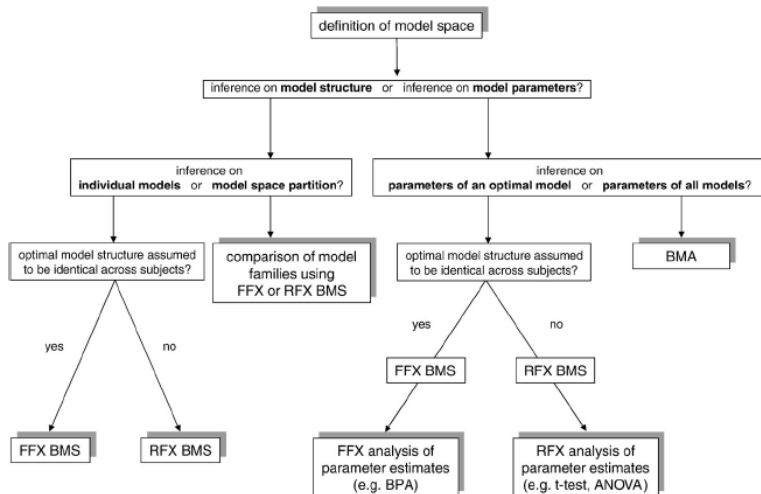
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Ten Simple Rules



Stephan et al. Neuroimage, 2010