Chapter 3

Solving the BEM equation

Contents

3.1 Approximation of the BEM equation

3.1.1 Original formulation

The previous chapter demonstrated how to express, equation 2.20, the forward problem as an integral form of Maxwell's equations :

$$V(\vec{s}) = V_\infty(\vec{s}) - \frac{1}{2\pi}\sum_{l}^{N_S}\frac{\sigma_l^- - \sigma_l^+}{\sigma_k^- + \sigma_k^+}\int_{S_l'} V(\vec{s}')\vec{\nabla}'\left(\frac{1}{|\vec{s}-\vec{s}'|}\right)\vec{n}(\vec{s}')\,dS_l' \qquad (3.1)$$

where $V_\infty(\vec{s})$ is the potential due to $\vec{j}_f$ in a conductor of infinite extent and homogeneous conductivity $(\sigma_k^- + \sigma_k^+)/2$:

$$V_\infty(\vec{s}) = \frac{1}{2\pi(\sigma_k^- + \sigma_k^+)} \int_v \vec{j}_f(\vec{r}\,') \vec{\nabla}' \left(\frac{1}{|\vec{s} - \vec{r}\,'|} \right) dv \tag{3.2}$$

and

- the sum $\sum_l^{N_S}$ runs over all the surfaces separating volumes of homogeneous isotropic conductivity.
- σ_l^- and σ_l^+ are the conductivity inside and outside[1] the surface S_l.
- $\vec{s}$ and $\vec{s}\,'$ are points on the surfaces S_k and S_l respectively.
- $\vec{n}(\vec{s}\,')$ is a unit vector normal to the surface S_l at the point $\vec{s}\,'$ and oriented from the inside towards the outside of S_l.

Solving equation 3.1 for V would provide us with a way to estimate the potential at any location on any surface given any source $\vec{j}_f$ in the brain volume.

3.1.2 General approach

The main task of solving the forward problem is to evaluate accurately the integrals on the right hand side of equation 3.1. The volume integral over the continuous sources distribution $\vec{j}_f$ can be easily calculated by approximating $\vec{j}_f$ as a superposition of independent point sources of known location and orientation. This reduces the volume integral into a sum of independent contributions for each location and orientation of the sources.

On the other hand the surface integrals are more difficult to calculate: they run on different and irregular surfaces and, moreover, they involve the potential $V(\vec{s})$ that is sought after. Therefore it will be necessary to express the surface integrals in terms of the value of the unknown function V at some discrete set of points on the surfaces, and to tessellate the surfaces into sets of regular patches.

The most obvious approximation for the surfaces is to model each of them by a set of plane triangles. With this surface tessellation, the surface integrals of 3.1 can be expressed as a sum of integrals over triangles:

$$V(\vec{s}) = V_\infty(\vec{s}) - \frac{1}{2\pi} \sum_{l=1}^{N_S} \frac{\sigma_l^- - \sigma_l^+}{\sigma_k^- + \sigma_k^+} \sum_{m=1}^{N_{tr}^{(l)}} \int_{\triangle_m^{(l)}} V(\vec{s}\,') \vec{\nabla}' \left(\frac{1}{|\vec{s} - \vec{s}\,'|} \right) \vec{n}(\vec{s}\,')\, dS' \tag{3.3}$$

where the surface S_l has been modelled by a set of $N_{tr}^{(l)}$ triangles $\triangle_m^{(l)}$.

The function V is rendered discrete by choosing on which nodal points V is evaluated and how the function V behaves on each individual plane triangle. This would allow an explicit calculation of the integrals over the triangles and equation 3.3 could eventually be simplified into a sum of known or, at least, easily evaluated analytical functions.

[1] the notion of "inside" and "outside" depends on the orientation of the normal $\vec{n}(\vec{s}\,')$ to the surface

Three different approximations of V over a triangle are usually considered. First, one could choose to evaluate V at the centre of gravity of each triangle and consider this value constant over the triangle, one value is thus obtained for each triangle. This approximation shall be referred as the "Centre of Gravity" (or "CoG") method (Hämäläinen & Sarvas, 1989; Meijs *et al.*, 1989). The function V could also be evaluated on the vertices of the triangles, this is generally called a "vertex" approximation (one value per vertex). If the potential over the triangle is supposed to be constant and equal to the mean of the potential at its vertices, this approximation will be called the "Constant Potential at Vertices" (or "CPV") method. On the other hand, if the potential is considered to be varying linearly over the triangle, this approximation will be named the "Linear Potential at Vertices" (or "LPV") method (Schlitt *et al.*, 1995).

The CoG and vertices (CPV and LPV) methods differ mainly on the choice of the nodal points where the unknown potential function V is calculated. It is important to note that for a closed tessellated surface there are about twice as many triangles as vertices. The number and arrangement of the triangles determine how well the true surface is spatially approximated. The choice of the potential approximation method determines the number of equations to be solved (one per triangle or vertex) and how well the true potential is modelled over each triangle (constant or linear approximation).

3.1.3 Current source model

In equation 3.2, the source function $\vec{j}_f(\vec{r}\,)$ is a continuous function throughout the volume. To be able to solve numerically the BEM equation 3.1, a discrete approximation of the source function has to be adopted. The source function $\vec{j}_f(\vec{r}\,)$ can be approximated by a distribution of N_j independent dipole sources of known location $\vec{r}_i$:

$$\vec{j}_f(\vec{r}\,) = \sum_{i=1}^{N_j} \left[\int_{v_i} \vec{j}_f(\vec{r}\,')\, dv_i \right] \delta(\vec{r} - \vec{r}_i) \tag{3.4a}$$

$$= \sum_{i=1}^{N_j} \vec{j}_f(\vec{r}_i)\, \delta(\vec{r} - \vec{r}_i) \tag{3.4b}$$

where $\vec{j}_f(\vec{r}_i) = \int_{v_i} \vec{j}_f(\vec{r}\,')\, dv_i$ is the summed activity in the volume v_i around the location $\vec{r}_i$ and $\delta(\vec{r}\,)$ is the discrete Dirac delta function[2].

Now with relation 3.4, equation 3.2 becomes :

$$V_\infty(\vec{s}\,) = \frac{1}{2\pi(\sigma_k^- + \sigma_k^+)} \int_v \vec{j}_f(\vec{r}\,')\vec{\nabla}'\!\left(\frac{1}{|\vec{s} - \vec{r}\,'|}\right) dv \tag{3.5a}$$

$$= \frac{1}{2\pi(\sigma_k^- + \sigma_k^+)} \int_v \frac{\vec{s} - \vec{r}\,'}{|\vec{s} - \vec{r}\,'|^3} \sum_{i=1}^{N_j} \vec{j}_f(\vec{r}_i)\, \delta(\vec{r}\,' - \vec{r}_i)\, dv \tag{3.5b}$$

$$= \frac{1}{2\pi(\sigma_k^- + \sigma_k^+)} \sum_{i=1}^{N_j} \vec{j}_f(\vec{r}_i) \int_v \frac{\vec{s} - \vec{r}\,'}{|\vec{s} - \vec{r}\,'|^3}\, \delta(\vec{r}\,' - \vec{r}_i)\, dv \tag{3.5c}$$

[2] $\delta(\vec{r}\,)$ equals 1 if $\vec{r} = \vec{o}$ and 0 for $\vec{r} \neq \vec{o}$

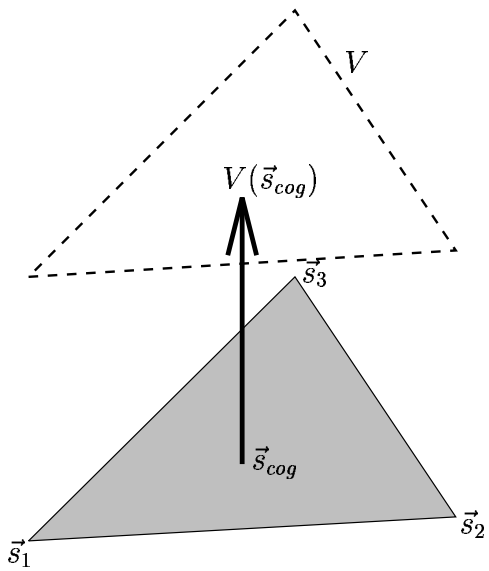

Figure 3.1: The Centre of Gravity (CoG) potential approximation: the potential V over the triangle is assumed to be constant and equal to the potential at the centre of gravity $\vec{s}_{cog}$ of the triangle, $V = V(\vec{s}_{cog})$.

and eventually we obtain:

$$V_\infty(\vec{s}) = \frac{1}{2\pi(\sigma_k^- + \sigma_k^+)} \sum_{i=1}^{N_j} \frac{\vec{s} - \vec{r}_i}{|\vec{s} - \vec{r}_i|^3} \, \vec{j}_f(\vec{r}_i) \tag{3.6}$$

3.1.4 Potential function model

The "Centre of Gravity" approximation

With this approximation, the unknown function V is calculated on nodal points located at the centre of gravity of each triangle. The potential over the triangle is supposed to be constant and equal to the potential at the centre of gravity $V = V(\vec{s}_{cog})$, as shown in figure 3.1.

With this approximation of the potential, the integral over each triangle in 3.3 can be simplified:

$$\begin{aligned} \int_{\triangle_m^{(l)}} V(\vec{s}\,') \vec{\nabla}' \left(\frac{1}{|\vec{s} - \vec{s}\,'|} \right) \vec{n}(\vec{s}\,') \, dS' &= V(\vec{s}\,'_{cog}) \int_{\triangle_m^{(l)}} \vec{\nabla}' \left(\frac{1}{|\vec{s} - \vec{s}\,'|} \right) \vec{n}(\vec{s}\,') \, dS' && \text{(3.7a)} \\ &= -V(\vec{s}\,'_{cog}) \, \Omega^{(l,m)}(\vec{s}) && \text{(3.7b)} \end{aligned}$$

where $\Omega^{(l,m)}(\vec{s})$ is the solid angle at $\vec{s}$ subtended by the triangle $\triangle_m^{(l)}$:

$$\begin{aligned} \Omega^{(l,m)}(\vec{s}) &= -\int_{\triangle_m^{(l)}} \vec{\nabla}' \left(\frac{1}{|\vec{s} - \vec{s}\,'|} \right) \vec{n}(\vec{s}\,') \, dS' && \text{(3.8a)} \\ &= \int_{\triangle_m^{(l)}} \frac{\vec{s}\,' - \vec{s}}{|\vec{s}\,' - \vec{s}|^3} \vec{n}(\vec{s}\,') \, dS' && \text{(3.8b)} \end{aligned}$$

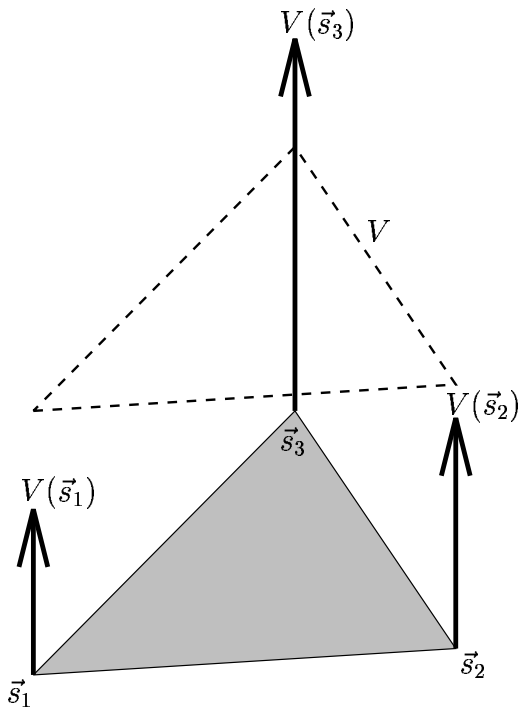

Figure 3.2: The Constant Potential at Vertices (CPV) potential approximation: The potential V over the triangle is assumed to be constant and equal to the mean of the potential at each vertex $\vec{s}_1$, $\vec{s}_2$ and $\vec{s}_3$ of the triangle, $V = (V(\vec{s}_1) + V(\vec{s}_2) + V(\vec{s}_3))/3$.

This last integral depends only on the three vector differences between $\vec{s}$ (the "point of view") and the three vertices $\vec{s}'_1$, $\vec{s}'_2$ and $\vec{s}'_3$ (the "points of support") determining the triangle $\triangle_m^{(l)}$. There exists an explicit analytic formula to calculate $\Omega^{(l,m)}(\vec{s})$, it will be presented in section 3.2.1.

The BEM equation 3.3 eventually becomes a "simple sum of known analytical functions" :

$$V(\vec{s}_{cog,p}) = V_\infty(\vec{s}_{cog,p}) + \frac{1}{2\pi} \sum_{l=1}^{N_S} \frac{\sigma_l^- - \sigma_l^+}{\sigma_k^- + \sigma_k^+} \sum_{m=1}^{N_{tr}^{(l)}} V(\vec{s}_{cog,m})\, \Omega^{(l,m)}(\vec{s}_{cog,p}) \tag{3.9}$$

where $\vec{s}_{cog,m}$ (resp. $\vec{s}_{cog,p}$) is the "centre of gravity" of the m^{th} (resp. p^{th}) triangle $\triangle_m^{(l)}$ (resp. $\triangle_p^{(k)}$) of the l^{th} (resp. k^{th}) surface S_l (resp. S_k). The BEM problem has now the form of a set of linear equations. Its solution is presented in section 3.3.

The "Constant Potential at Vertices" approximation

Here the potential is evaluated on the vertices of the triangles and the potential over each triangle is assumed to be constant and equal to the mean of the potential at its vertices $V = (V(\vec{s}_1) + V(\vec{s}_2) + V(\vec{s}_3))/3$, as shown in figure 3.2.

With this approximation of the potential, the integral over each triangle in 3.3 can be

simplified like this:

$$\int_{\triangle_m^{(l)}} V(\vec{s}\,')\vec{\nabla}'\Big(\frac{1}{|\vec{s}-\vec{s}\,'|}\Big)\,\vec{n}(\vec{s}\,')\,dS'$$

$$= \frac{V(\vec{s}\,'_1)+V(\vec{s}\,'_2)+V(\vec{s}\,'_3)}{3}\int_{\triangle_m^{(l)}}\vec{\nabla}'\Big(\frac{1}{|\vec{s}-\vec{s}\,'|}\Big)\,\vec{n}(\vec{s}\,')\,dS' \quad (3.10a)$$

$$= -\frac{V(\vec{s}\,'_1)+V(\vec{s}\,'_2)+V(\vec{s}\,'_3)}{3}\,\Omega^{(l,m)}(\vec{s}) \quad (3.10b)$$

where $\Omega^{(l,m)}(\vec{s})$ is the solid angle at $\vec{s}$ subtended by the triangle $\triangle_m^{(l)}$, expressed like in equation 3.8. An analytical formula to calculate it is provided in section 3.2.1.

The BEM equation 3.3 also becomes a "simple sum of known analytical functions":

$$V(\vec{s}_\bullet) = V_\infty(\vec{s}_\bullet) + \frac{1}{2\pi}\sum_{l=1}^{N_S}\frac{\sigma_l^- - \sigma_l^+}{\sigma_k^- + \sigma_k^+}\sum_{m=1}^{N_{tr}^{(l)}}\frac{V(\vec{s}\,'_{1,m})+V(\vec{s}\,'_{2,m})+V(\vec{s}\,'_{3,m})}{3}\,\Omega^{(l,m)}(\vec{s}_\bullet) \quad (3.11)$$

where $\vec{s}_\bullet$ is one of the three vertices of a triangle of the k^{th} surface S_k and $\vec{s}_{i,m}{}'$ is the i^{th} vertex of the m^{th} triangle $\triangle_m^{(l)}$ of the l^{th} surface S_l. The BEM problem is now also expressed as a set of linear equations, its solution is presented in section 3.3.

The "Linear Potential at Vertices" approximation

Here the potential is also evaluated on the vertices of the triangles but a better approximation of the potential over the triangles is used: The potential is assumed to be varying linearly over each triangle, as shown in figure 3.3.

As only three values are needed to specify a linear function on a plane surface, the value of the potential V at the three vertices of the triangle can be used. Moreover this ensures that the potential varies continuously from one triangle to the next which was not the case with the two previous approximations.

As for the CPV approximation, the integral over each triangle in equation 3.3 can be simplified into a weighted sum of the potential at the vertices:

$$\int_{\triangle_m^{(l)}} V(\vec{s}\,')\vec{\nabla}'\Big(\frac{1}{|\vec{s}-\vec{s}\,'|}\Big)\,\vec{n}(\vec{s}\,')\,dS'$$

$$= -\Big(V(\vec{s}\,'_1)\,\Omega_1^{(l,m)}(\vec{s}) + V(\vec{s}\,'_2)\,\Omega_2^{(l,m)}(\vec{s}) + V(\vec{s}\,'_3)\,\Omega_3^{(l,m)}(\vec{s})\Big) \quad (3.12)$$

The three $\Omega_\bullet^{(l,m)}(\vec{s})$ are also purely geometric quantities depending on the vector differences between the "point of view" $\vec{s}$ and the vertices $\vec{s}\,'_\bullet$ of the triangle. An explicit analytical formula to calculate the $\Omega_\bullet^{(l,m)}(\vec{s})$ from $\vec{s}$ and $\vec{s}\,'_\bullet$ is presented in section 3.2.2.

With this approximation the BEM equation 3.3 also becomes a "simple sum of known analytical functions":

$$V(\vec{s}_\bullet) = V_\infty(\vec{s}_\bullet) + \quad (3.13)$$

$$\frac{1}{2\pi}\sum_{l=1}^{N_S}\frac{\sigma_l^- - \sigma_l^+}{\sigma_k^- + \sigma_k^+}\sum_{m=1}^{N_{tr}^{(l)}}\Big(V(\vec{s}\,'_{1,m})\,\Omega_1^{(l,m)}(\vec{s}_\bullet) + V(\vec{s}\,'_{2,m})\,\Omega_2^{(l,m)}(\vec{s}_\bullet) + V(\vec{s}\,'_{3,m})\,\Omega_3^{(l,m)}(\vec{s}_\bullet)\Big)$$

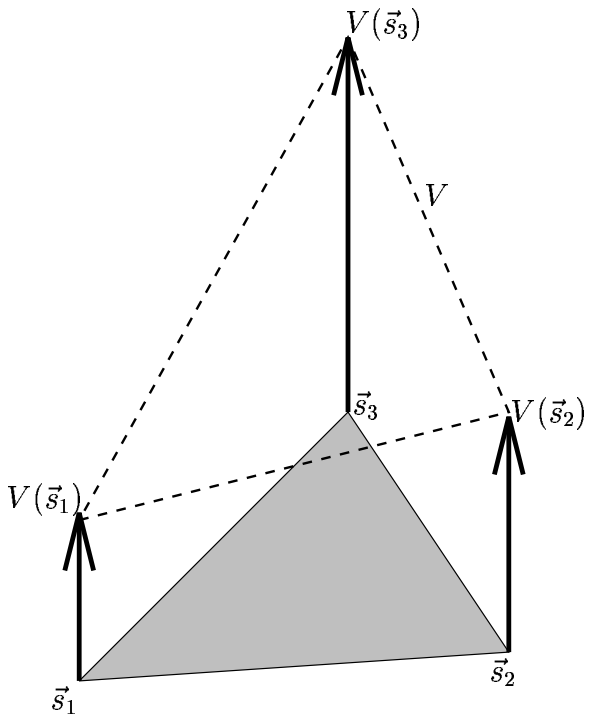

Figure 3.3: The Linear Potential at Vertices (LPV) potential approximation: The potential V over the triangle is assumed to be varying linearly between the potential calculated at each vertex $\vec{s}_1$, $\vec{s}_2$ and $\vec{s}_3$ of the triangle, $V = f(V(\vec{s}_1), V(\vec{s}_2), V(\vec{s}_3))$.

where $\vec{s}_\bullet$ is one of the three vertices of a triangle of the k^{th} surface S_k and $\vec{s}_{i,m}{}'$ is the i^{th} vertex of the m^{th} triangle $\triangle_m^{(l)}$ of the l^{th} surface S_l. The BEM problem is also reduced to a set of linear equations, and its solution is presented in section 3.3.

3.2 Solid angle calculation

3.2.1 Constant potential approximation

For both CoG and CPV approximations presented in section 3.1.4, the solid angle $\Omega^{(l,m)}(\vec{s})$ subtended by a plane triangle $\triangle_m^{(l)}$ at some point $\vec{s}$ has to be calculated.

Without loss of generality, the observation point $\vec{s}$ can be placed at the origin $\vec{o}^*$. The three vertices $\vec{s}_1$, $\vec{s}_2$ and $\vec{s}_3$ of the plane triangle are then specified by the vectors $\vec{v}_1 = \vec{s}_1 - \vec{o}^*$, $\vec{v}_2 = \vec{s}_2 - \vec{o}^*$ and $\vec{v}_3 = \vec{s}_3 - \vec{o}^*$ relative to this origin $\vec{o}^*$, as shown in figure 3.4. The solid angle Ω can be analytically expressed with $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ by the formula taken from van Oosterom & Strackee (1983):

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{\vec{v}_1(\vec{v}_2 \times \vec{v}_3)}{|\vec{v}_1|\,|\vec{v}_2|\,|\vec{v}_3| + (\vec{v}_1\vec{v}_2)|\vec{v}_3| + (\vec{v}_1\vec{v}_3)|\vec{v}_2| + (\vec{v}_2\vec{v}_3)|\vec{v}_1|} \tag{3.14}$$

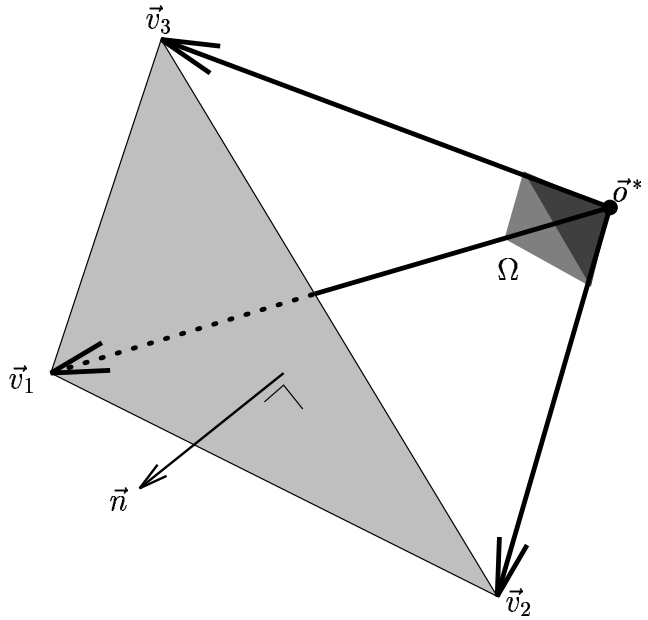

Figure 3.4: Solid angle supported by a plane triangle: The solid angle Ω supported at the point $\vec{o}^*$ by the plane triangle (grey shade) depends only on the three vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ and can be easily calculated by equation 3.14.

For the case where the point of view $\vec{s}$ is located on the same plane as the triangle $\triangle_m^{(l)}$, e.g. when $\vec{s}$ is at the centre of gravity of the triangle in the CoG approximation or when $\vec{s}$ is at a vertex of the triangle in the CPV approximation, the solid angle $\Omega^{(l,m)}(\vec{s})$ is zero. The consequences of this feature are presented in section 3.2.3.

3.2.2 Linear potential approximation

For the LPV approximation presented in section 3.1.4, three geometric quantities Ω_i ($i = 1, 2, 3$) have to be calculated for each triangle, under the assumption that the potential V varies linearly over this triangle. As for the previous section, the observation point $\vec{s}$ can again be placed at the origin $\vec{o}^*$ without loss of generality. The three vertices $\vec{s}_1$, $\vec{s}_2$ and $\vec{s}_3$ of the plane triangle are then specified by the vectors $\vec{v}_1 = \vec{s}_1 - \vec{o}^*$, $\vec{v}_2 = \vec{s}_2 - \vec{o}^*$ and $\vec{v}_3 = \vec{s}_3 - \vec{o}^*$ relative to this origin $\vec{o}^*$, as shown in figure 3.4. There also exists an analytical formula for the Ω_i (de Munck, 1992; Schlitt *et al.*, 1995):

$$\Omega_i = \frac{1}{2A}\left(\vec{z}_i\vec{n}\,\Omega + \beta(\vec{v}_j - \vec{v}_k)\vec{\Omega}\right) \tag{3.15}$$

where

- A is the surface of the plane triangle,
- $\vec{z}_i = \vec{v}_j \times \vec{v}_k$ with (i, j, k) a cyclic permutation of $(1, 2, 3)$,
- $\vec{n}$ is a unit vector normal to the triangle,

- Ω is the solid angle subtended by the plane triangle at the origin as expressed in formula 3.14,
- $\beta = \vec{n}\vec{v}_i$ is equal to the perpendicular distance from the origin to the triangle,
- $\vec{\Omega}$ is a vector defined by $\vec{\Omega} = \sum_{i=1}^{3}(\gamma_j - \gamma_i)\vec{v}_j$ with

$$\gamma_i = \frac{1}{|\vec{v}_j - \vec{v}_i|} \ln\left(\frac{|\vec{v}_j - \vec{v}_i|\,|\vec{v}_j| + (\vec{v}_j - \vec{v}_i)\vec{v}_j}{|\vec{v}_j - \vec{v}_i|\,|\vec{v}_i| + (\vec{v}_j - \vec{v}_i)\vec{v}_i}\right) \tag{3.16}$$

The Ω_i also satisfy the equality: $\Omega_1 + \Omega_2 + \Omega_3 = \Omega$. When the triangle $\triangle_m^{(l)}$ and the point of view $\vec{s}$ are coplanar, for example when $\vec{s}$ is at one of the vertices of the triangle, Ω and β are equal to zero, so $\Omega_1 = \Omega_2 = \Omega_3 = 0$. This problem is tackled in section 3.2.3.

3.2.3 The auto-solid angle problem

An important property of solid angle concerns its integral over a single closed surface. We know from equation 3.8 that the infinitesimal solid angle $d\Omega'$ subtended by the infinitesimal surface dS' around the point $\vec{s}'$ at the point of view $\vec{s}$ is expressed by:

$$d\Omega'(\vec{s}, \vec{s}') = \vec{n}(\vec{s}')\frac{\vec{s}' - \vec{s}}{|\vec{s}' - \vec{s}|^3}\, dS' \tag{3.17}$$

then the integral of $d\Omega'(\vec{s}, \vec{s}')$ over a smooth closed surface is equal to:

$$\Omega_S(\vec{s}) = \int_S d\Omega'(\vec{s}, \vec{s}') = \begin{bmatrix} 0 \\ 2\pi \\ 4\pi \end{bmatrix}, \text{ for } \vec{s} \begin{bmatrix} \text{outside} \\ \text{on} \\ \text{inside} \end{bmatrix} \text{ the surface.} \tag{3.18}$$

The BEM equation 3.1 contains an integral of the form:

$$\int_S d\Omega'(\vec{s}, \vec{s}')V(\vec{s}') \tag{3.19}$$

then this integral is converted into a discrete sum, by equation 3.9, 3.11 or 3.14, of the form:

$$\sum_{m=1}^{M} \Omega_{nm}\, V_m \tag{3.20}$$

where:

- m and n are indices of nodal points (possibly on different surfaces) where the potential V is calculated.
- V_m is the potential at the point m.
- Ω_{nm} is the solid angle associated with the point m for the point of view n.
- M is the number of nodal points on the surface, i.e. the number of triangles for the CoG approximation or the number of vertices for the CPV and LPV approximations.

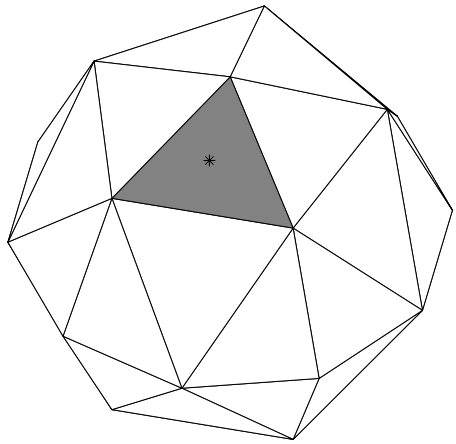

Figure 3.5: The auto-solid angle problem for the CoG approximation: The solid angle subtended by the grey triangle from its centre of gravity (the black dot) is zero but the total solid angle subtended by the rest of the surface (white triangles) is equal to 2π.

It is therefore important that the Ω_{nm} satisfy the relation 3.18:

$$\Omega_{n,S} = \sum_{m=1}^{M} \Omega_{nm} = \begin{bmatrix} 0 \\ 2\pi \\ 4\pi \end{bmatrix} \text{, for } \vec{s} \begin{bmatrix} \text{outside} \\ \text{on} \\ \text{inside} \end{bmatrix} \text{ the surface.} \tag{3.21}$$

In the case of $\vec{s}$ being outside or inside the surface, there is no problem to satisfy these equalities: All the Ω_{nm} can be unambiguously calculated with equation 3.14 or 3.15. But when $\vec{s}$ is on the surface itself then we meet the "auto-solid angle problem": the solid angle subtended by a triangle which contains the point of view is zero and the second equality of relation 3.21 may not be satisfied automatically.

The auto-solid angle problem for the CoG approximation

In the CoG approximation, as the potential is evaluated on the "centre of gravity" of each triangle, there will be only one null solid angle: $\Omega_{mm} = 0$, as can be seen in figure 3.5. Since the rest of the solid angle subtended by the closed surface is already 2π, there is no missing angle and the second equality of the relations 3.21 is satisfied.

Nevertheless, in reality the surface modelled by the triangle is not plane, and it should thus support some non-zero solid angle. There is no way to improve the solution, but to use a finer meshing of the surface.

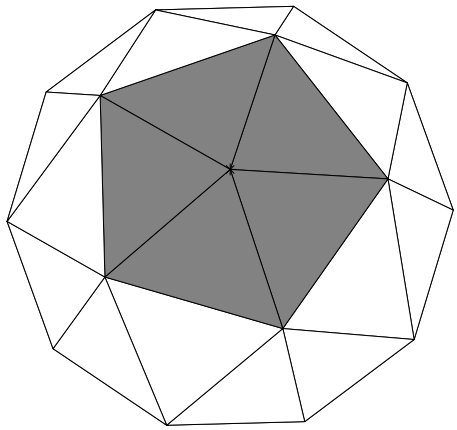

Figure 3.6: The auto-solid angle problem for the CPV and LPV approximations : The solid angle supported by the "central" point and the adjacent grey triangles is zero, therefore the total solid angle supported by the remaining white triangles is less than 2π.

The auto-solid angle problem for the CPV approximation

In the "vertices" approximations (CPV and LPV), all the adjacent triangles (grey triangles in figure 3.6) containing the "point of view" are supporting a null solid angle. The solid angle Ω_r subtended by the rest of the surface will **not** be equal to 2π because the adjacent triangles do not represent a flat surface.

$$\Omega_{miss} = 2\pi - \Omega_r = 2\pi - \sum_{m=1}^{M} \Omega_{nm} \tag{3.22}$$

It is necessary to distribute this "missing solid angle" Ω_{miss} over the triangles adjacent to the "point of view".

In the case of the CPV approximation, the easiest way to proceed is to share uniformly this "missing solid angle" between the adjacent triangles and their vertices. The central "point of view" should be attributed one third of the missing solid angle $\Omega_{miss}/3$, and the other vertices should receive $2\Omega_{miss}/(3N_{adj})$, where N_{adj} is the number of triangles adjacent to the central "point of view".

There exists a more elaborate way to split the missing solid angle between the adjacent triangles. The approach is based on a local spherical approximation of the surface surrounding the central vertex and is presented in the next section, in relation with the LPV approximation.

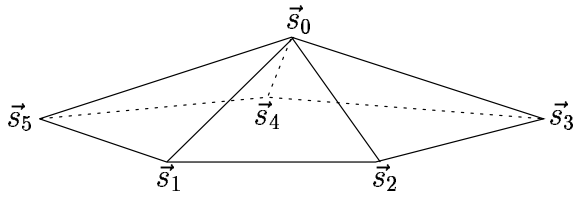

Figure 3.7: Adjacent triangles supporting a non-zero solid angle: The surface defined by $\vec{s}_1$, $\vec{s}_2$, ...,$\vec{s}_5$ around $\vec{s}_0$ support a non-zero solid angle. Each triangle, defined by a triplet of vertices ($[\vec{s}_0\vec{s}_1\vec{s}_2]$, $[\vec{s}_0\vec{s}_2\vec{s}_3]$, ..., $[\vec{s}_0\vec{s}_5\vec{s}_1]$), supports a part of the missing solid angle which in turn must be shared between its vertices. Each triangle can be locally approximated by a portion of sphere.

The auto-solid angle problem for the LPV approximation

The case of the LPV approximation is more complicated because the potential V is varying linearly over each triangle. There are two main problems: how to divide up the missing the solid angle Ω_{miss} between the adjacent triangles and, within each of them, how to share its part between its vertices. This is illustrated in figure 3.7. There exists an analytical formula to solve these problems (Heller, 1990) but it requires that each triangle around the point of view $\vec{s}_0$ be approximated by a portion of sphere of centre $\vec{r}_c$ and radius R. If the surface is regular and smooth compared to the density of the mesh, this local spherical approximation will hold as R will be much larger than the length of the edges of the triangles.

Since three points do not determine a sphere, a fourth point must be chosen. A suitable point would be the next adjacent vertex, e.g. the sphere that passes through the triplet $[\vec{s}_0\vec{s}_1\vec{s}_2]$ could be required to pass through $\vec{s}_3$ as well. A better and more anatomically correct approximation can be obtained, if, at the tessellation stage, the centre of gravity $\vec{s}_{cog}$ of each triangle is projected perpendicular to the triangular plane onto the actual surface of the volume $\vec{s}_{cog}^{\perp}$. Then a sphere can easily be fitted through these four points: the three vertices defining the triangle and its projected centre of gravity, as shown in figure 3.8. This approach was used in the implementation of the BEM solution.

Once the spheres have been fitted for $\vec{s}_0$ and its adjacent vertices $\vec{s}_1$, $\vec{s}_2$, ..., $\vec{s}_{N_{adj}}$, an approximate value for the solid angle subtended by each triplet $[\vec{s}_0\vec{s}_1\vec{s}_2]$, $[\vec{s}_0\vec{s}_2\vec{s}_3]$, ..., $[\vec{s}_0\vec{s}_{N_{adj}}\vec{s}_1]$ at $\vec{s}_0$ can be calculated. Using spherical coordinates for the vertices, as shown in figure 3.9, the solid angle $\Omega_{[\vec{s}_0\vec{s}_1\vec{s}_2]}$ subtended at $\vec{s}_0$ by the spherical region bounded by $\vec{s}_0$, $\vec{s}_1$ and $\vec{s}_2$, is approximated by:

$$\Omega_{[\vec{s}_0\vec{s}_1\vec{s}_2]} = \frac{\psi_1 + \psi_2}{4}\phi_{12} \tag{3.23}$$

where ψ_1 and ψ_2 are easily obtained and

$$\sin\frac{\phi_{12}}{2} = \frac{|\vec{s}_b - \vec{s}_2|}{2R\sin\psi_2} \tag{3.24}$$

with

$$\vec{s}_b = \vec{r}_c + \frac{1}{\sin\psi_1}[\sin(\psi_1 - \psi_2)(\vec{s}_0 - \vec{r}_c) + \sin\psi_2(\vec{s}_1 - \vec{r}_c)] \tag{3.25}$$

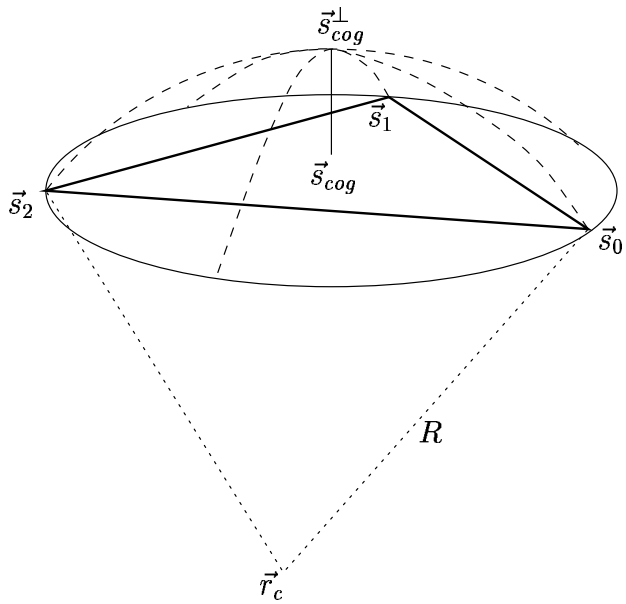

Figure 3.8: Spherical approximation of an adjacent plane triangle: The triplet of vertices $[\vec{s}_0\vec{s}_1\vec{s}_2]$ and the projection $\vec{s}^{\perp}_{cog}$ of the centre of gravity $\vec{s}_{cog}$ of the triangle on the actual surface determine a sphere, centre $\vec{r}_c$ and radius R, that approximates the actual surface.

After calculation of the solid angle for all the other triplets and summation, the total solid angle $\Omega_{\vec{s}_0}$ supported by $\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_{N_{adj}}$ at $\vec{s}_0$ is approximated by:

$$\Omega_{\vec{s}_0} = \Omega_{[\vec{s}_0\vec{s}_1\vec{s}_2]} + \Omega_{[\vec{s}_0\vec{s}_2\vec{s}_3]} + \ldots + \Omega_{[\vec{s}_0\vec{s}_{N_{adj}}\vec{s}_1]} \tag{3.26}$$

The fraction of missing solid angle Ω_{miss} to be assigned to each triangle, e.g., $[\vec{s}_0\vec{s}_1\vec{s}_2]$, is obtained by:

$$f_{[\vec{s}_0\vec{s}_1\vec{s}_2]} = \frac{\Omega_{\vec{s}_0}}{\Omega_{[\vec{s}_0\vec{s}_1\vec{s}_2]}} \tag{3.27}$$

Note that, even though approximations are made in these calculations, since they only involve ratios, the total solid angle subtended by the region around $\vec{s}_0$ will sum to Ω_{miss}, and the total solid angle subtended by the entire surface at $\vec{s}_0$ will be exactly 2π. For the CPV approximation, the portion $f_{[\vec{s}_0\vec{s}_1\vec{s}_2]}\Omega_{miss}$ of missing solid angle is simply equally distributed between the three vertices of the triangle. For the LPV approximation, it is necessary to further share this portion of missing solid angle between the vertices of the adjacent triangles.

Assuming that ψ_1, ψ_2 and ϕ_{12} are small and that the potential V varies linearly with the distance on the sphere, Heller (1990) showed that it is possible to share the solid angle $f_{[\vec{s}_0\vec{s}_1\vec{s}_2]}\Omega_{miss}$ between the three vertices $\vec{s}_0$, $\vec{s}_1$ and $\vec{s}_2$ such that:

$$\Omega_{\vec{s}_0,[\vec{s}_0\vec{s}_1\vec{s}_2]} + \Omega_{\vec{s}_1,[\vec{s}_0\vec{s}_1\vec{s}_2]} + \Omega_{\vec{s}_2,[\vec{s}_0\vec{s}_1\vec{s}_2]} = f_{[\vec{s}_0\vec{s}_1\vec{s}_2]}\Omega_{miss} \tag{3.28}$$

where

$$\Omega_{\vec{s}_0,[\vec{s}_0\vec{s}_1\vec{s}_2]} = \frac{\phi_{12}}{48}\left(7\psi_1 + 7\psi_2 - \frac{\psi_1^2}{\psi_2} - \frac{\psi_2^2}{\psi_1}\right)\frac{\Omega_{miss}}{\Omega_{\vec{s}_0}} \tag{3.29a}$$

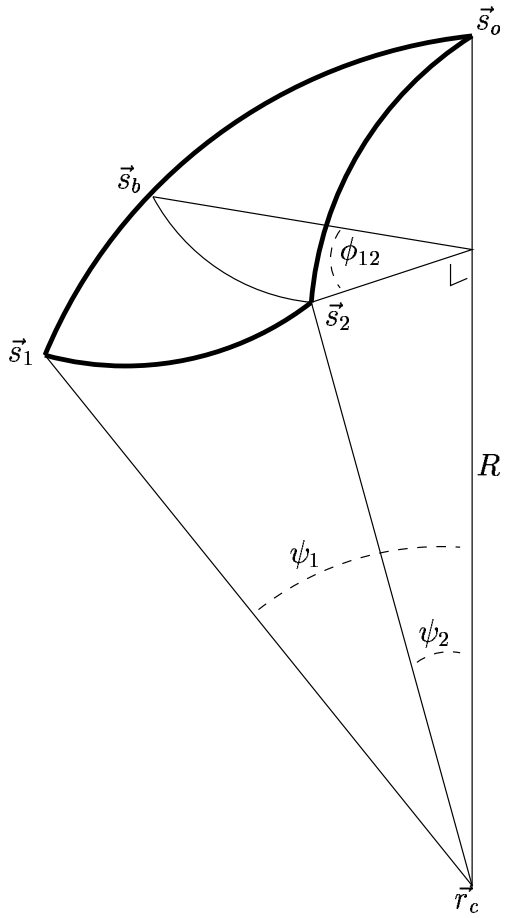

Figure 3.9: Autosolid angle approximation: The triangle defined by the triplet $[\vec{s}_0\vec{s}_1\vec{s}_2]$ is approximated by a portion of a sphere. The solid angle subtended at $\vec{s}_0$ by the curved surface (bold line) can be calculated using the spherical coordinates of $\vec{s}_1$ and $\vec{s}_2$: ψ_1, ψ_2 and ϕ_{12}.

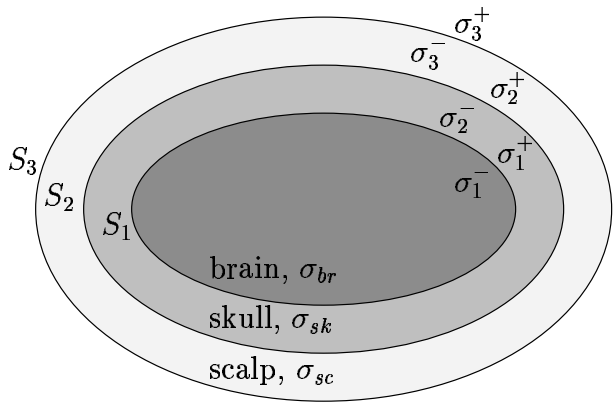

Figure 3.10: Simplified realistic head model: Three concentric volumes of isotropic conductivity, brain (σ_{br}), skull (σ_{sk}) and scalp (σ_{sc}), separated by the three surfaces S_1, S_2 and S_3, are used as head model.

$$\Omega_{\vec{s}_1,[\vec{s}_0\vec{s}_1\vec{s}_2]} = \frac{\phi_{12}}{48}\left(3\psi_1 + 2\psi_2 + \frac{\psi_2^2}{\psi_1}\right)\frac{\Omega_{miss}}{\Omega_{\vec{s}_0}} \tag{3.29b}$$

$$\Omega_{\vec{s}_2,[\vec{s}_0\vec{s}_1\vec{s}_2]} = \frac{\phi_{12}}{48}\left(2\psi_1 + 3\psi_2 + \frac{\psi_1^2}{\psi_2}\right)\frac{\Omega_{miss}}{\Omega_{\vec{s}_0}} \tag{3.29c}$$

These approximations for the CPV and LPV auto-solid angle were employed in the implementation of the BEM solution.

3.3 Matrix form of the BEM equation

3.3.1 Simple realistic head model

For the simplified head model, three concentric volumes of homogeneous conductivity are considered: the brain, skull and scalp volumes, of respective conductivity σ_{br}, σ_{sk} and σ_{sc}, as depicted in figure 3.10. The three interfaces: "brain-skull", "skull-scalp" and "scalp-air" separating the three volumes are numbered respectively 1, 2 and 3. With this numbering convention, the conductivity inside and outside each surface is defined by:

$$\begin{array}{rcccl} & & \sigma_1^- & = & \sigma_{br} \\ \sigma_1^+ & = & \sigma_2^- & = & \sigma_{sk} \\ \sigma_2^+ & = & \sigma_3^- & = & \sigma_{sc} \\ \sigma_3^+ & = & & & 0 \end{array} \tag{3.30}$$

3.3.2 Matrix form

With the discrete approximation of the source term (equation 3.6) and the approximation of the boundary element equation: CoG approximation (equation 3.9), CPV approximation (equation 3.11) or LPV approximation (equation 3.14) adopted in section 3.2, the

BEM problem can be expressed under a matrix form:

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{B}_{13} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \mathbf{B}_{23} \\ \mathbf{B}_{31} & \mathbf{B}_{32} & \mathbf{B}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \\ \mathbf{G}_3 \end{bmatrix} [\mathbf{j}] \tag{3.31}$$

or, in a reduced version,

$$\mathbf{v} = \mathbf{B}\,\mathbf{v} + \mathbf{G}\,\mathbf{j} \tag{3.32}$$

where:

- $\mathbf{v}_k$, a $N_{v_k} \times 1$ vector, contains the value of the potential at the N_{v_k} nodal points of surface S_k: centre of gravity of each triangle for the CoG approximation or vertices of the triangles for the CPV and LPV approximations.
- $\mathbf{B}_{kl}$, a $N_{v_k} \times N_{v_l}$ matrix, represents the influence of the potential of surface S_l on the potential of surface S_k. Its elements depend on the conductivity inside and outside the surfaces S_k and S_l, and on the solid angles used in the BEM approximations 3.9. $\mathbf{B}$ is a $N_v \times N_v$ matrix with $N_v = N_{v_1} + N_{v_2} + N_{v_3}$.
- $\mathbf{j} = [\,\vec{j}_1{}^t\ \vec{j}_2{}^t\ \dots\ \vec{j}_{N_j}{}^t\,]^t$, a $3N_j \times 1$ vector, is the source distribution vector where each $\vec{j}_n = [\,j_{n,x}\ j_{n,y}\ j_{n,z}\,]^t$ is an orientation-free source vector.
- $\mathbf{G}_k$, a $N_{v_k} \times 3N_j$ matrix, is the free space potential matrix depending on the location $\vec{r}_n$ of the sources $\vec{j}_n$, the nodal points on surface S_k and the conductivity inside and outside surface S_k (σ_k^- and σ_k^+). $\mathbf{G}$ is a $N_v \times 3N_j$ matrix.

Self influence matrix B

For the CoG approximation, the element (p, q) of the matrix $\mathbf{B}_{kl}$ is calculated by:

$$\mathbf{B}_{kl}^{(p,q)} = \frac{1}{2\pi}\left(\frac{\sigma_l^- - \sigma_l^+}{\sigma_k^- + \sigma_k^+}\right)\Omega_{pq} \tag{3.33}$$

where

- p (resp. q) is the index of the nodal point on the surface S_k (resp. S_l).
- Ω_{pq} is the solid angle at the centre of gravity of the p^{th} triangle of S_k subtended by the q^{th} triangle of S_l.

For the CPV approximation, the element (p, q) of the matrix $\mathbf{B}_{kl}$ is calculated by:

$$\mathbf{B}_{kl}^{(p,q)} = \frac{1}{2\pi}\left(\frac{\sigma_l^- - \sigma_l^+}{\sigma_k^- + \sigma_k^+}\right)\sum_n^{N_q}\frac{\Omega_{pn}}{3} \tag{3.34}$$

where

- p (resp. q) is the index of the nodal point on the surface S_k (resp. S_l).
- N_q is the number of triangles comprising the q^{th} vertex.
- Ω_{pn} is the solid angle at the p^{th} vertex of the surface S_k subtended by the n^{th} triangle (containing the q^{th}vertex) of S_l.

For the LPV approximation, the element (p, q) of the matrix $\mathbf{B}_{kl}$ is calculated by:

$$\mathbf{B}_{kl}^{(p,q)} = \frac{1}{2\pi}\left(\frac{\sigma_l^- - \sigma_l^+}{\sigma_k^- + \sigma_k^+}\right)\sum_n^{N_q}\Omega_{pn}^q \tag{3.35}$$

where

- p (resp. q) is the index of the nodal point on the surface S_k (resp. S_l).
- N_q is the number of triangles comprising the q^{th} vertex.
- Ω_{pn}^q is the portion of solid angle attributed to the q^{th} vertex, and subtended by the n^{th} triangle (containing the q^{th} vertex) of S_l at the p^{th} vertex of the surface S_k.

Note that the correction for the aut-solid angle problem can only be performed **after** calculating all the solid angles relative to one nodal point because the missing solid angle Ω_{miss} is estimated by equation 3.22. When Ω_{miss} is calculated, then the procedure presented in section 3.2.3 can easily be applied for the CPV and LPV approximations.

Free potential matrix $\mathbf{G}_k$

The elements $(p, 3q-2)$, $(p, 3q-1)$ and $(p, 3q)$ of the matrix $\mathbf{G}_k$ can be calculated with:

$$\left[\mathbf{G}_k^{(p,3q-2)}\ \mathbf{G}_k^{(p,3q-1)}\ \mathbf{G}_k^{(p,3q)}\right] = \frac{(\vec{s}_p - \vec{r}_q)^t}{2\pi(\sigma_k^- + \sigma_k^+)|\vec{s}_p - \vec{r}_q|^3} \tag{3.36}$$

where

- $\vec{s}_p$ is the p^{th} nodal point of the surface S_k.
- $\vec{r}_q$ is the location of the q^{th} current source $\vec{j}_q$.

Interpretation

The $N_v \times N_v$ matrix $\mathbf{B}$ can be interpreted as a "self influence" matrix: the potential at any nodal point on any surface is influenced by the potential at all the other nodal points on all the surfaces. The $N_v \times 3N_j$ matrix $\mathbf{G}$ is, on the contrary, the "direct influence" matrix: in free space, the potential at any nodal point on any surface depends only on the source distribution $\mathbf{j}$. The matrices $\mathbf{B}$ and $\mathbf{G}$ can be calculated separately and they depend solely on the geometry of the problem and the conductivity adopted for each volume. Any change in the geometry or the conductivity implies the recalculation of these matrices.

The number and placement of triangles determines how well the true surface is approximated by the set of plane triangles. The choice of the potential approximation on each triangle determines how well the real potential is approximated but also how the matrices $\mathbf{B}$ and $\mathbf{G}$ are calculated and therefore the number of equations to be solved. For a tessellated closed surface there are about twice as many triangles than vertices. Thus with the

same head model, there will be twice as many equations to solve if the CoG approximation is used than with the CPV or LPV approximation. The LPV approximation is also the only one that ensures a continuous variation of the electric potential from one triangle to the next on the surfaces.

3.4 Solving the matrix BEM equation

The solution of the forward problem consists now in establishing a linear relationship between the source distribution $\mathbf{j}$ and the potential on the surfaces $\mathbf{v}$ (or at least on the scalp $\mathbf{v}_3$) of the form:

$$\mathbf{v} = \mathbf{L}\,\mathbf{j} \tag{3.37}$$

An obvious solution of equation 3.32 would be to simply solve the system of equations:

$$(\mathbf{I}_{N_v} - \mathbf{B})\,\mathbf{v} = \mathbf{G}\,\mathbf{j} \tag{3.38}$$

by inverting the matrix $(\mathbf{I}_{N_v} - \mathbf{B})$. However we are dealing here with a problem of electric potential and a potential function can only be measured relative to some reference point, i.e. calculated to within a constant. The systems of equations 3.38 is therefore rank deficient and the matrix $(\mathbf{I}_{N_v} - \mathbf{B})$ **cannot** be inverted.

As $\mathbf{v}_a = \mathbf{v}$ and $\mathbf{v}_b = \mathbf{v} + c\,\mathbf{1}_{N_v}$ (with $c \neq 0$) must both satisfy 3.32 and 3.38, it follows:

$$\left.\begin{array}{rcl} \mathbf{v} & = & \mathbf{B}\,\mathbf{v} + \mathbf{G}\,\mathbf{j} \\ (\mathbf{v} + c\,\mathbf{1}_{N_v}) & = & \mathbf{B}\,(\mathbf{v} + c\,\mathbf{1}_{N_v}) + \mathbf{G}\,\mathbf{j}, \quad c \neq 0 \end{array}\right\} \begin{array}{l} \Rightarrow \quad c\,\mathbf{1}_{N_v} = \mathbf{B}\,c\,\mathbf{1}_{N_v}, \quad c \neq 0 \\ \Rightarrow \quad \mathbf{B}\,\mathbf{1}_{N_v} = \mathbf{1}_{N_v} \\ \Rightarrow \quad (\mathbf{I}_{N_v} - \mathbf{B})\,\mathbf{1}_{N_v} = 0 \end{array}$$

The matrix $(\mathbf{I}_{N_v} - \mathbf{B})$ has a **nu**ll eigenvalue associated with the eigenvector $\mathbf{1}_{N_v}$, i.e $\mathbf{B}$ has a **unit** eigenvalue associated with the eigenvector $\mathbf{1}_{N_v}$. The only way to solve 3.38 is to use a "deflation technique" (Lynn & Timlake, 1968a,b; Chan, 1984).

3.4.1 Deflation technique

By assuming that the unit eigenvalue of $\mathbf{B}$ is simple, it can easily be shown that any other solution will only differ by an additive constant, that is, a scalar multiple of $\mathbf{1}_{N_v}$. Let $\mathbf{p}$ be any vector such that $\mathbf{1}_{N_v}^t\,\mathbf{p} = 1$ and suppose that we seek the solution of 3.32 such that $\mathbf{p}^t\,\mathbf{v} = 0$. Then looking for this particular solution, equation 3.32 becomes:

$$\mathbf{v} = (\mathbf{B} - \mathbf{1}_{N_v}\,\mathbf{p}^t)\,\mathbf{v} + \mathbf{G}\,\mathbf{j} \tag{3.39}$$

Under the assumption that $\mathbf{p}^t\,\mathbf{v} = 0$, the matrix $\mathbf{C} = (\mathbf{B} - \mathbf{1}_{N_v}\,\mathbf{p}^t)$ is a deflation of $\mathbf{B}$ and has no unit eigenvalue, so that $(\mathbf{I}_{N_v} - \mathbf{C})^{-1} = (\mathbf{I}_{N_v} - \mathbf{B} - \mathbf{1}_{N_v}\,\mathbf{p}^t)^{-1}$ exists. Equation 3.31 can be rewritten like

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13} \\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23} \\ \mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{C}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \\ \mathbf{G}_3 \end{bmatrix} [\mathbf{j}] \tag{3.40}$$

and this system of equations can be solved by calculating:

$$\mathbf{v} = (\mathbf{I}_{N_v} - \mathbf{C})^{-1}\,\mathbf{G}\,\mathbf{j} \tag{3.41a}$$
$$= (\mathbf{I}_{N_v} - \mathbf{B} - \mathbf{1}_{N_v}\,\mathbf{p}^t)^{-1}\,\mathbf{G}\,\mathbf{j} \tag{3.41b}$$

where $\mathbf{v}$ will satisfy $\mathbf{p}^t\,\mathbf{v} = 0$.

Each vector $\mathbf{v}_\bullet$ is of size $N_{v_\bullet} \times 1$, so if, for example, $\mathbf{p}$ is defined by:

$$\mathbf{p} = [\underbrace{0\ 0\ \ldots\ 0}_{N_{v_1}}\ \underbrace{0\ 0\ \ldots\ 0}_{N_{v_2}}\ \underbrace{p\ p\ \ldots\ p}_{N_{v_3}}]^t \tag{3.42}$$

with $p = 1/N_{v_3}$, then $\mathbf{p}^t\,\mathbf{v} = 0$ simply means that the mean of $\mathbf{v}_3$ is zero. Therefore equation 3.41 provides us with the solution that is mean corrected over the scalp surface.

3.4.2 Partial solution for the scalp

The number of equations (N_v) to solve in 3.40 is rather large, but only the direct relationship between the source distribution $\mathbf{j}$ and the potential on the scalp $\mathbf{v}_3$ is of interest in the EEG problem. After some algebraic manipulations, equation 3.40 can be rewritten like this:

$$\mathbf{\Gamma}_1\,\mathbf{v}_3 = \mathbf{\Gamma}_2\,\mathbf{j} \tag{3.43}$$

where

$$\begin{aligned}\mathbf{\Gamma}_1 = {} & -(\mathbf{C}_{33} - \mathbf{I}_{N_{v_3}}) \\ & -\mathbf{C}_{31}\left[(\mathbf{C}_{11} - \mathbf{I}_{N_{v_1}}) - \mathbf{C}_{12}(\mathbf{C}_{22} - \mathbf{I}_{N_{v_2}})^{-1}\mathbf{C}_{21}\right]^{-1}\left[\mathbf{C}_{12}(\mathbf{C}_{22} - \mathbf{I}_{N_{v_2}})^{-1}\mathbf{C}_{23} - \mathbf{C}_{13}\right] \\ & -\mathbf{C}_{32}\left[(\mathbf{C}_{22} - \mathbf{I}_{N_{v_2}}) - \mathbf{C}_{21}(\mathbf{C}_{11} - \mathbf{I}_{N_{v_1}})^{-1}\mathbf{C}_{12}\right]^{-1}\left[\mathbf{C}_{21}(\mathbf{C}_{11} - \mathbf{I}_{N_{v_1}})^{-1}\mathbf{C}_{13} - \mathbf{C}_{23}\right]\end{aligned} \tag{3.44}$$

and

$$\begin{aligned}\mathbf{\Gamma}_2 = {} & \mathbf{G}_3 \\ & +\mathbf{C}_{31}\left[(\mathbf{C}_{11} - \mathbf{I}_{N_{v_1}}) - \mathbf{C}_{12}(\mathbf{C}_{22} - \mathbf{I}_{N_{v_2}})^{-1}\mathbf{C}_{21}\right]^{-1}\left[\mathbf{C}_{12}(\mathbf{C}_{22} - \mathbf{I}_{N_{v_2}})^{-1}\mathbf{G}_2 - \mathbf{G}_1\right] \\ & +\mathbf{C}_{32}\left[(\mathbf{C}_{22} - \mathbf{I}_{N_{v_2}}) - \mathbf{C}_{21}(\mathbf{C}_{11} - \mathbf{I}_{N_{v_1}})^{-1}\mathbf{C}_{12}\right]^{-1}\left[\mathbf{C}_{21}(\mathbf{C}_{11} - \mathbf{I}_{N_{v_1}})^{-1}\mathbf{G}_1 - \mathbf{G}_2\right]\end{aligned} \tag{3.45}$$

In order to reduce the number of operations required to calculate $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ in 3.44 and 3.45, these relations can be expressed using some common "intermediate matrices" $\mathbf{\Upsilon}_i$:

$$\mathbf{\Gamma}_1 = -\left((\mathbf{C}_{33} - \mathbf{I}_{N_{v_3}}) + \mathbf{\Upsilon}_5\,\mathbf{C}_{13} + \mathbf{\Upsilon}_6\,\mathbf{C}_{23}\right) \tag{3.46a}$$
$$\mathbf{\Gamma}_2 = \mathbf{G}_3 + \mathbf{\Upsilon}_5\,\mathbf{G}_1 + \mathbf{\Upsilon}_6\,\mathbf{G}_2 \tag{3.46b}$$

where

$$\mathbf{\Upsilon}_6 = \mathbf{\Upsilon}_4\,\mathbf{\Upsilon}_2 - \mathbf{\Upsilon}_3 \tag{3.47a}$$
$$\mathbf{\Upsilon}_5 = \mathbf{\Upsilon}_3\,\mathbf{\Upsilon}_1 - \mathbf{\Upsilon}_4 \tag{3.47b}$$
$$\mathbf{\Upsilon}_4 = \mathbf{C}_{31}\left(-\mathbf{\Upsilon}_2\,\mathbf{C}_{21} + (\mathbf{C}_{11} - \mathbf{I}_{N_{v_1}})\right)^{-1} \tag{3.47c}$$
$$\mathbf{\Upsilon}_3 = \mathbf{C}_{32}\left(-\mathbf{\Upsilon}_1\,\mathbf{C}_{12} + (\mathbf{C}_{22} - \mathbf{I}_{N_{v_2}})\right)^{-1} \tag{3.47d}$$
$$\mathbf{\Upsilon}_2 = \mathbf{C}_{12}\left(\mathbf{C}_{22} - \mathbf{I}_{N_{v_2}}\right)^{-1} \tag{3.47e}$$
$$\mathbf{\Upsilon}_1 = \mathbf{C}_{21}\left(\mathbf{C}_{11} - \mathbf{I}_{N_{v_1}}\right)^{-1} \tag{3.47f}$$

By proceeding carefully, one has only to solve 4 systems of equations (3.47f, 3.47e, 3.47d and 3.47c) to obtain $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$. The matrices to inverse are only of size N_{v_1} and N_{v_2}, thus the calculation of $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ require much less computational effort than inverting directly $\mathbf{C}$, which is of size $N_v = N_{v_1} + N_{v_2} + N_{v_3}$.

The explicit solution for all nodal points on the scalp surface may then be obtained from 3.43 by calculating:

$$\mathbf{v}_3 = \mathbf{\Gamma}_1^{-1}\mathbf{\Gamma}_2\mathbf{j} \tag{3.48a}$$
$$= \mathbf{L}\mathbf{j} \tag{3.48b}$$

Where $\mathbf{\Gamma}_1$ is only of size N_{v_3}. It would of course be possible to obtain a relation such as 3.43 for the other two surfaces.

It is important to note that $\mathbf{\Gamma}_1$ depends **only** on the matrix $\mathbf{C}$, i.e. on the geometry and the conductivity of the volumes, but not on the source distribution $\mathbf{j}$. By pre-calculating and saving $\mathbf{\Gamma}_1^{-1}$, $\mathbf{\Upsilon}_5$ and $\mathbf{\Upsilon}_6$, the lead field matrix $\mathbf{L}$ can be obtained very easily for any source distribution, using equations 3.46b and 3.48.

3.4.3 Partial solution for the electrode sites

In general it is not necessary to calculate the potential V over the entire scalp surface as, in practical cases, EEG is recorded from a limited number of electrodes. Therefore the lead field for the electrode sites should only be calculated:

$$\mathbf{v}_{3,el} = \mathbf{L}_{el}\mathbf{j} \tag{3.49}$$

In a realistic head model, the location of the electrodes is defined relative to the triangular mesh of the scalp. As the electrodes have typically a diameter of a few millimetres, the location of the electrodes can be approximated to the triangle directly underneath. If the CoG approximation was used to model the potential over each triangle, the nodal points for which the lead field should be calculated, are simply the centre of gravity of the triangles under the electrodes. If the CPV or LPV approximation was used, the potential is estimated at the vertices of each triangle. Therefore the lead field for the three vertices of the electrode triangles has to be calculated, and should be combined afterwards to provide a single lead field per electrode. If the electrodes are about the same size as the triangles on the scalp, then a single mean can be used. But, if the electrodes are smaller than the triangles, a linear interpolation between the vertices of the triangle should be preferred.

A partial solution of equation 3.43 for a few nodal points is possible thanks to the Frobenius-Schur formula that allows the partial calculation of the inverse of a matrix:

$$\begin{bmatrix} \mathbf{M} & \mathbf{N} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{M}^{-1} + \mathbf{M}^{-1}\mathbf{N}\,\mathbf{F}^{-1}\mathbf{P}\,\mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{N}\,\mathbf{F}^{-1} \\ -\mathbf{F}^{-1}\mathbf{P}\,\mathbf{M}^{-1} & \mathbf{F}^{-1} \end{bmatrix} \tag{3.50}$$

where: $\begin{cases} \mathbf{M} \text{ and } \mathbf{Q} \text{ must be square.} \\ \mathbf{M} \text{ and } \mathbf{F} = \mathbf{Q} - \mathbf{P}\,\mathbf{M}^{-1}\mathbf{N} \text{ must be invertible.} \end{cases}$

Considering that the N_{el} interesting (respectively, N_{ot} other) nodal points are the last N_{el} (respectively, first N_{ot}) elements of $\mathbf{v}_3$: $\mathbf{v}_3 = [\mathbf{v}_{3,ot}^t \; \mathbf{v}_{3,el}^t]^t$, then equation 3.43 can be rewritten like this:

$$\underbrace{\left[\begin{array}{c|c} \mathbf{M} & \mathbf{N} \\ \hline \mathbf{P} & \mathbf{Q} \end{array}\right]}_{\mathbf{\Gamma}_1} \underbrace{\left[\begin{array}{c} \mathbf{v}_{3,ot} \\ \hline \mathbf{v}_{3,el} \end{array}\right]}_{\mathbf{v}_3} = \underbrace{\left[\begin{array}{c} \mathbf{R} \\ \hline \mathbf{S} \end{array}\right]}_{\mathbf{\Gamma}_2} \left[\begin{array}{c} \\ \mathbf{j} \\ \\ \end{array}\right] \tag{3.51}$$

$$\Rightarrow \underbrace{\left[\begin{array}{c} \mathbf{v}_{3,ot} \\ \hline \mathbf{v}_{3,el} \end{array}\right]}_{\mathbf{v}_3} = \underbrace{\left[\begin{array}{c|c} \mathbf{M}^{-1} + \mathbf{M}^{-1}\mathbf{N}\,\mathbf{F}^{-1}\mathbf{P}\,\mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{N}\,\mathbf{F}^{-1} \\ \hline -\mathbf{F}^{-1}\mathbf{P}\,\mathbf{M}^{-1} & \mathbf{F}^{-1} \end{array}\right]}_{\mathbf{\Gamma}_1^{-1}} \underbrace{\left[\begin{array}{c} \mathbf{R} \\ \hline \mathbf{S} \end{array}\right]}_{\mathbf{\Gamma}_2} \left[\begin{array}{c} \\ \mathbf{j} \\ \\ \end{array}\right] \tag{3.52}$$

This partitioning of the vertices is not natural (as the electrodes are spread over the scalp surface) but such ordering may be easily obtained by adequately permuting the rows and columns in $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$.

As only the bottom part of equation 3.52 is of interest, the lead field for the electrode sites can be obtained from the submatrices of $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ by:

$$\mathbf{L}_{el} = \left[\begin{array}{cc} -\mathbf{F}^{-1}\mathbf{P}\,\mathbf{M}^{-1} & \mathbf{F}^{-1} \end{array}\right] \left[\begin{array}{c} \mathbf{R} \\ \mathbf{S} \end{array}\right] \tag{3.53}$$

and only the two matrices $\mathbf{F}$ and $\mathbf{M}$ have to be inverted. The number N_{el} of electrode sites being much smaller than the number of nodal points on the scalp surface N_{v_3}, the time spent to calculate $\mathbf{F}^{-1}$ is negligible compared to $\mathbf{M}^{-1}$.

By using the simplified expression 3.46b for $\mathbf{\Gamma}_2$, equation 3.53 becomes:

$$\begin{aligned} \mathbf{L}_{el} &= \left[\begin{array}{cc} -\mathbf{F}^{-1}\mathbf{P}\,\mathbf{M}^{-1} & \mathbf{F}^{-1} \end{array}\right] (\mathbf{\Upsilon}_5\,\mathbf{G}_1 + \mathbf{\Upsilon}_6\,\mathbf{G}_2 + \mathbf{G}_3) && \text{(3.54a)} \\ &= \mathbf{\Xi}_1\,\mathbf{G}_1 + \mathbf{\Xi}_2\,\mathbf{G}_2 + \mathbf{\Xi}_3\,\mathbf{G}_3 && \text{(3.54b)} \end{aligned}$$

where

$$\begin{aligned} \mathbf{\Xi}_1 &= \left[\begin{array}{cc} -\mathbf{F}^{-1}\mathbf{P}\,\mathbf{M}^{-1} & \mathbf{F}^{-1} \end{array}\right] \mathbf{\Upsilon}_5 && \text{(3.55a)} \\ \mathbf{\Xi}_2 &= \left[\begin{array}{cc} -\mathbf{F}^{-1}\mathbf{P}\,\mathbf{M}^{-1} & \mathbf{F}^{-1} \end{array}\right] \mathbf{\Upsilon}_6 && \text{(3.55b)} \\ \mathbf{\Xi}_3 &= \left[\begin{array}{cc} -\mathbf{F}^{-1}\mathbf{P}\,\mathbf{M}^{-1} & \mathbf{F}^{-1} \end{array}\right] && \text{(3.55c)} \end{aligned}$$

The three matrices $\mathbf{\Xi}_1$, $\mathbf{\Xi}_2$ and $\mathbf{\Xi}_3$ depend **only** on the geometry and conductivity of the head model. If they are pre-calculated (and saved), the lead field $\mathbf{L}_{el}$ can be rapidly calculated for any source distribution $\mathbf{j}$ using equation 3.54. This is of particular interest if the location of the dipoles has to be modified, for example, if a denser mesh of dipoles is required in a linear distributed solution, or if an iterative procedure is used to optimise the location of the ECDs in a ECD-based solution.