The Normal, χ^2, t and F Probability Distributions

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Abstract

When we take functions of random variables the probability distribution changes. In this note I describe how to compute the new distribution using the 'method of transformations'. The procedure is applied to derive the χ^2, t and F distributions which are all defined in terms of simple functions of normal variables.

1 Introduction

When we take functions of random variables the probability distribution changes. In this note I describe how to compute the new distribution using the 'method of transformations'. The procedure is applied to derive the χ^2, t and F distributions which are all defined by taking simple functions of normal variables. These relationships are shown in Figure 1.

2 The method of transformations

Because probabilities are defined as areas under PDFs when we transform a variable

$$y = f(x) \tag{1}$$

we transform the PDF by preserving the areas

$$p(y)|dy| = p(x)|dx| \tag{2}$$

where the absolute value is taken because the changes in x or y (dx and dy) may be negative and areas must be positive. This is shown schematically in Figure 2. Hence

$$p(y) = \frac{p(x)}{|\frac{dy}{dx}|} \tag{3}$$

where the derivative is evaluated at $x = f^{-1}(y)$. This means that the function $f(x)$ must be one-to-one and invertible.

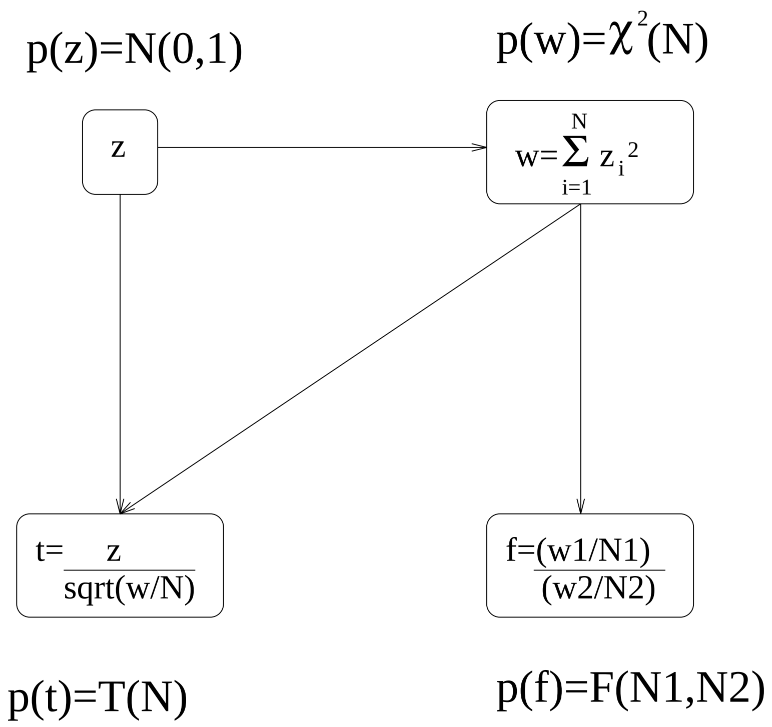

Figure 1: *The relationship between Normal, χ^2, t and F random variables*

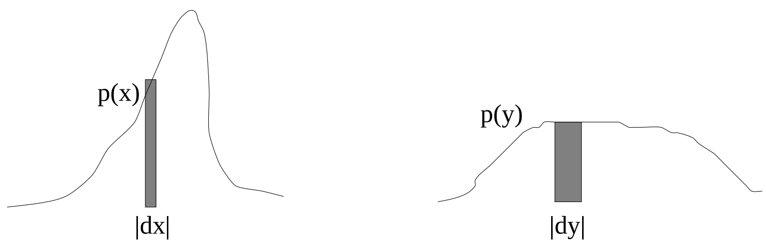

Figure 2: *Areas are preserved when transforming probabilities.*

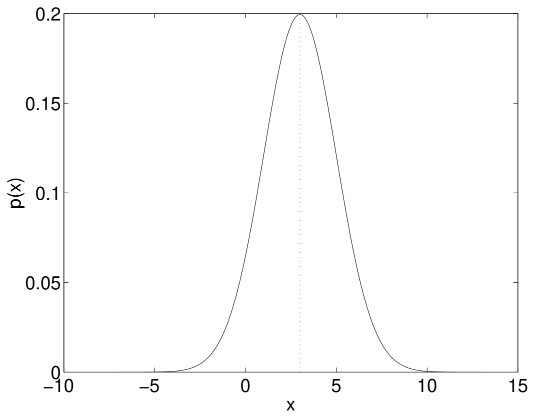

Figure 3: *The Gaussian Probability Density Function with* $\mu = 3$ *and* $\sigma = 2$.

For functions of two random variables eg. $y = f(x_1, x_2)$ we can (i) fix x_1 and write $p(y|x_1)$ as it is equivalent to $p(x_2)$ with the appropriate substitution for x_2 in terms of y and x_1, (ii) write down the joint density $p(y, x_1) = p(y|x_1)p(x_1)$ and then (iii) integrate out x_1 to get $p(y)$.That is

$$p(y) = \int p(y|x_1)p(x_1)dx_1 \tag{4}$$

3 The Normal distribution

The *Normal* or *Gaussian* probability density function, for the case of a single variable, is

$$p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{5}$$

where μ and σ^2 are the mean and variance. For a normalised variable

$$z = \frac{x-\mu}{\sigma} \tag{6}$$

we have

$$p(z) = \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{z^2}{2}\right) \tag{7}$$

4 The χ^2-distribution

If $z_1, z_2, ..., z_N$ are independent normally distributed random variables with mean zero and variance 1 then

$$w = \sum_{i=1}^{N} z_i^2 \tag{8}$$

has a χ^2-distribution with N degrees of freedom

$$p(w) = \chi^2(N) \tag{9}$$

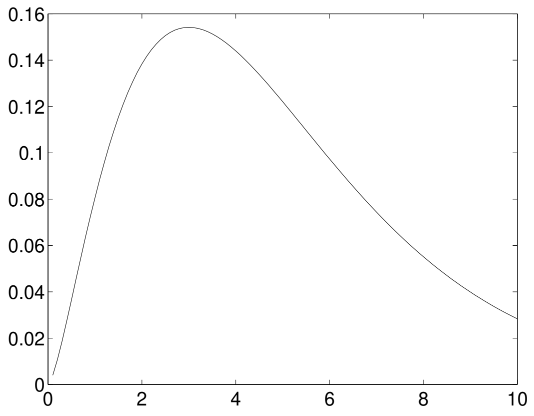

Figure 4: *The χ^2 Density for $N = 5$ degrees of freedom.*

This can be shown to be given by

$$p(w) = \frac{1}{\Gamma(N/2)} \frac{w^{N/2-1}}{2^{N/2}} \exp\left(\frac{-w}{2}\right) \tag{10}$$

The mean and variance are N and $2N$. The χ^2 distribution is only defined for positive variables.

We can derive the above distribution as follows. Firstly, consider the simple transformation $w_i = z_i^2$. We therefore have

$$p(w_i) = p(z_i) \left| \frac{dz_i}{dw_i} \right|_{z_i=\sqrt{w_i}} \tag{11}$$

This gives

$$p(w_i) = \frac{w_i^{-1/2}}{(2\pi)^{1/2}} \exp(-w_i/2) \tag{12}$$

which is the χ^2 distribution for $N = 1$ as $\Gamma(1/2) = \pi^{1/2}$ (we also note that because w is positive a factor of 2 disappears from the denominator).

For $N = 2$ we can write

$$p(w) = \int p(w|z_1)p(z_1)dz_1 \tag{13}$$

where

$$p(w|z_1) = p(z_2) \left| \frac{dz_2}{dw} \right| \tag{14}$$

evaluated at $z_2 = \sqrt{w - z_1^2}$. This gives

$$p(w) = \frac{1}{2\pi} \exp(-w/2) \int_{-w}^{w} \frac{1}{\sqrt{w - z_1^2}} dz_1 \tag{15}$$

As the indefinite integral gives $sin^{-1}(z_1/w)$, the definite integral equates to π, giving

$$p(w) = \frac{1}{2} \exp(-w/2) \tag{16}$$

which is equal to $\chi^2(N)$ for $N = 2$. We can derive the general result using

$$p(w) = \int p(w|z_1, z_2, ..., z_{N-1}) p(z_{N-1}) dz_{N-1} \tag{17}$$

5 The T-distribution

If $z_1, z_2, ..., z_N$ are independent standard normally distributed random variables and w is a χ^2 variable with N degrees of freedom (defined as in the previous section) then

$$t = \frac{z}{\sqrt{w/N}} \tag{18}$$

has a t-distribution with N degrees of freedom. We can derive the t-distribution using

$$p(t) = \int p(t, w) dw \tag{19}$$

where

$$p(t, w) = p(t|w) p(w) \tag{20}$$

We can write (from the previous definition of a χ^2 variable)

$$p(w) = \frac{1}{\Gamma(N/2)} \frac{w^{N/2-1}}{2^{N/2}} \exp\left(\frac{-w}{2}\right) \tag{21}$$

To calculate the conditional density we use

$$p(t|w) = p(z) \left|\frac{dz}{dt}\right| \tag{22}$$

where $z = \sqrt{w/N} t$. This gives

$$p(t|w) = \frac{w^{1/2}}{(2\pi)^{1/2} N^{1/2}} \exp\left(\frac{-wt^2}{2N}\right) \tag{23}$$

The joint density can then be integrated over to give

$$p(t) = \frac{1}{2^{(N+1)/2} \pi^{1/2} N^{1/2} \Gamma(N/2)} \int w^{(N-1)/2} \exp\left(\frac{-w}{2}(1 + t^2/N)\right) dw \tag{24}$$

Now, from the definition of a Gamma function we can write

$$\int_0^\infty x^{p-1} \exp(-ax) dx = a^{-p} \Gamma(p) \tag{25}$$

By equating $x = w$, $p - 1 = (N-1)/2$ and $a = (1 + t^2/N)/2$ we can use the above result to give

$$p(t) = \frac{\Gamma([N+1]/2)}{\pi^{1/2} N^{1/2} \Gamma(N/2)} (1 + t^2/N)^{-(N+1)/2} \tag{26}$$

which is the T-distribution.

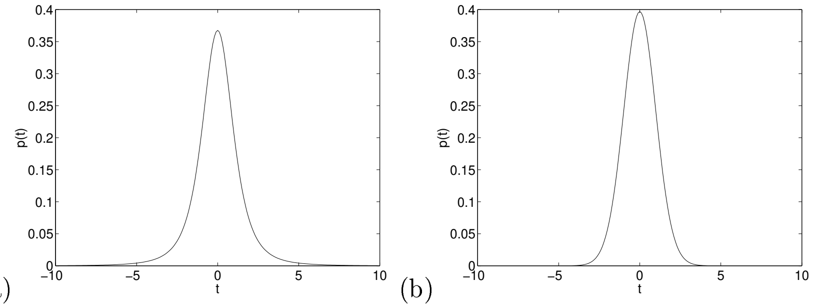

Figure 5: *The T-distribution with (a)* $N = 3$ *and (b)* $N = 49$ *degrees of freedom.*

6 The F-distribution

If w_1 and w_2 are independent χ^2 random variables with N_1 and N_2 degrees of freedom then

$$f = \frac{w_1/N_1}{w_2/N_2} \tag{27}$$

has an F-distribution with N_1 numerator and N_2 denominator degrees of freedom which we write as

$$p(f) = F(N_1, N_2) \tag{28}$$

We can derive the distribution using

$$p(f) = \int p(f|w_2)p(w_2)dw_2 \tag{29}$$

To calculate the conditional density we use

$$p(f|w_2) = p(w_1)\left|\frac{dw_1}{df}\right| \tag{30}$$

where $w_1 = w_2 N_1/N_2$. This gives

$$p(f|w_2) = \frac{1}{\Gamma(N_1/2)} \frac{w_2^{N_1/2}(N_1/N_2)^{N_1/2} f^{(N_1/2)-1}}{2^{N_1/2}} \exp\left(\frac{-w_2 N_1 f}{2N_2}\right) \tag{31}$$

Combining with $p(w_2)$ gives the joint density which can be integrated over as follows

$$p(f) = \frac{(N_1/N_2)^{N_1/2}}{\Gamma(N_1/2)\Gamma(N_2/2)} f^{(N_1/2)-1} \int w_2^{(N_1+N_2)/2-1} \exp\left[-\frac{w_2}{2}\left(1 + \frac{N_1 f}{N_2}\right)\right] dw_2 \tag{32}$$

Using the standard Gamma integral result quoted in the previous section we can equate $x = w_2$, $p = (N_1 + N_2)/2$ and $a = (1 + N_1 f/N_2)/2$ to give the F-distribution as

$$F(N_1, N_2) = \frac{\Gamma[(N_1+N_2)/2](N_1/N_2)^{N_1/2}}{\Gamma(N_1/2)\Gamma(N_2/2)} f^{(N_1/2)-1} \left(1 + \frac{N_1 f}{N_2}\right)^{-(N_1+N_2)/2} \tag{33}$$

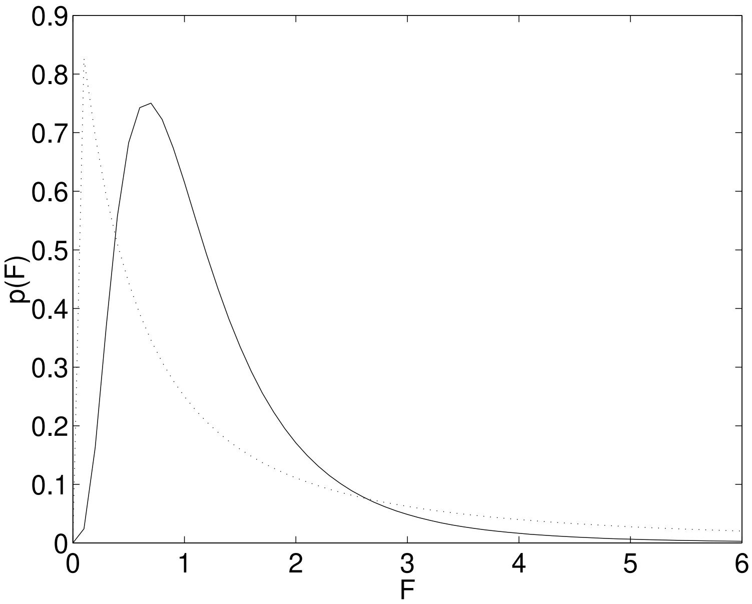

Figure 6: *The F-distribution with* $N_1 = N2 = 10$ *(solid line) and* $N_1 = N_2 = 2$ *(dotted line).*