Bayesian Inference

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Bayesian Inference

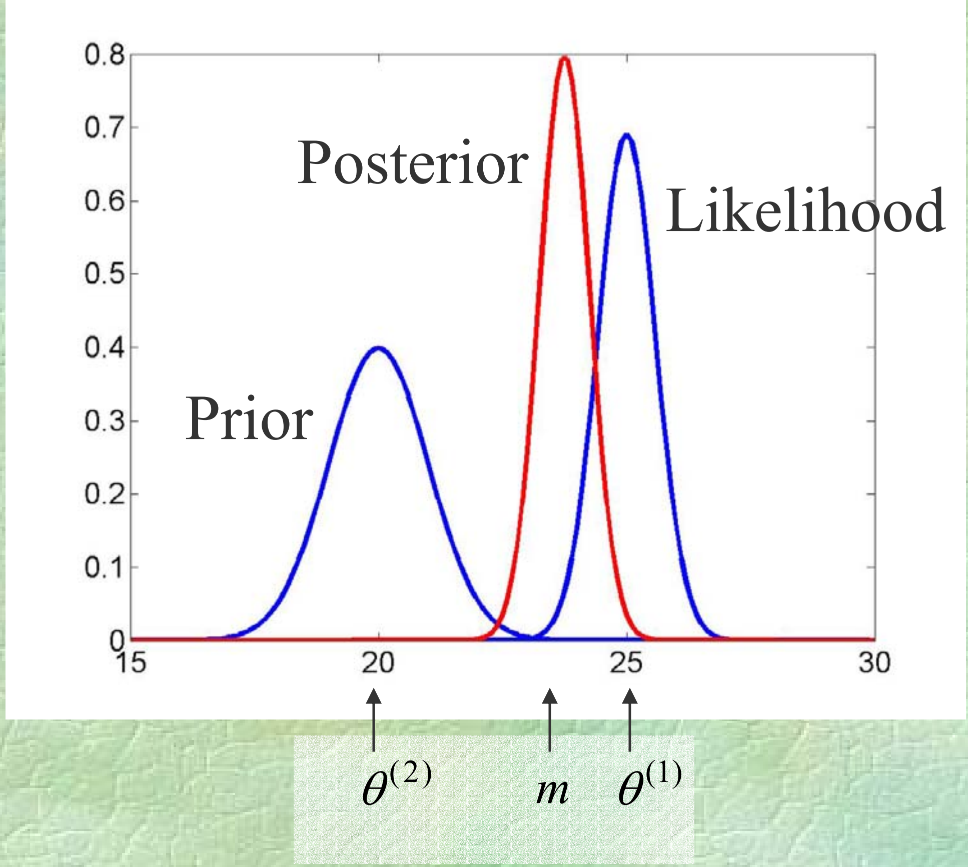
- Gaussians
- Posterior Probability Maps (PPMs)
- Hemodynamic Models (HDMs)
- Comparing models (DCMs)

One parameter

Likelihood and Prior

$$p(y \mid \theta^{(1)}) = N(\theta^{(1)}, \lambda^{(1)})$$

$$p(\theta^{(1)}) = N(\theta^{(2)}, \lambda^{(2)})$$



Posterior

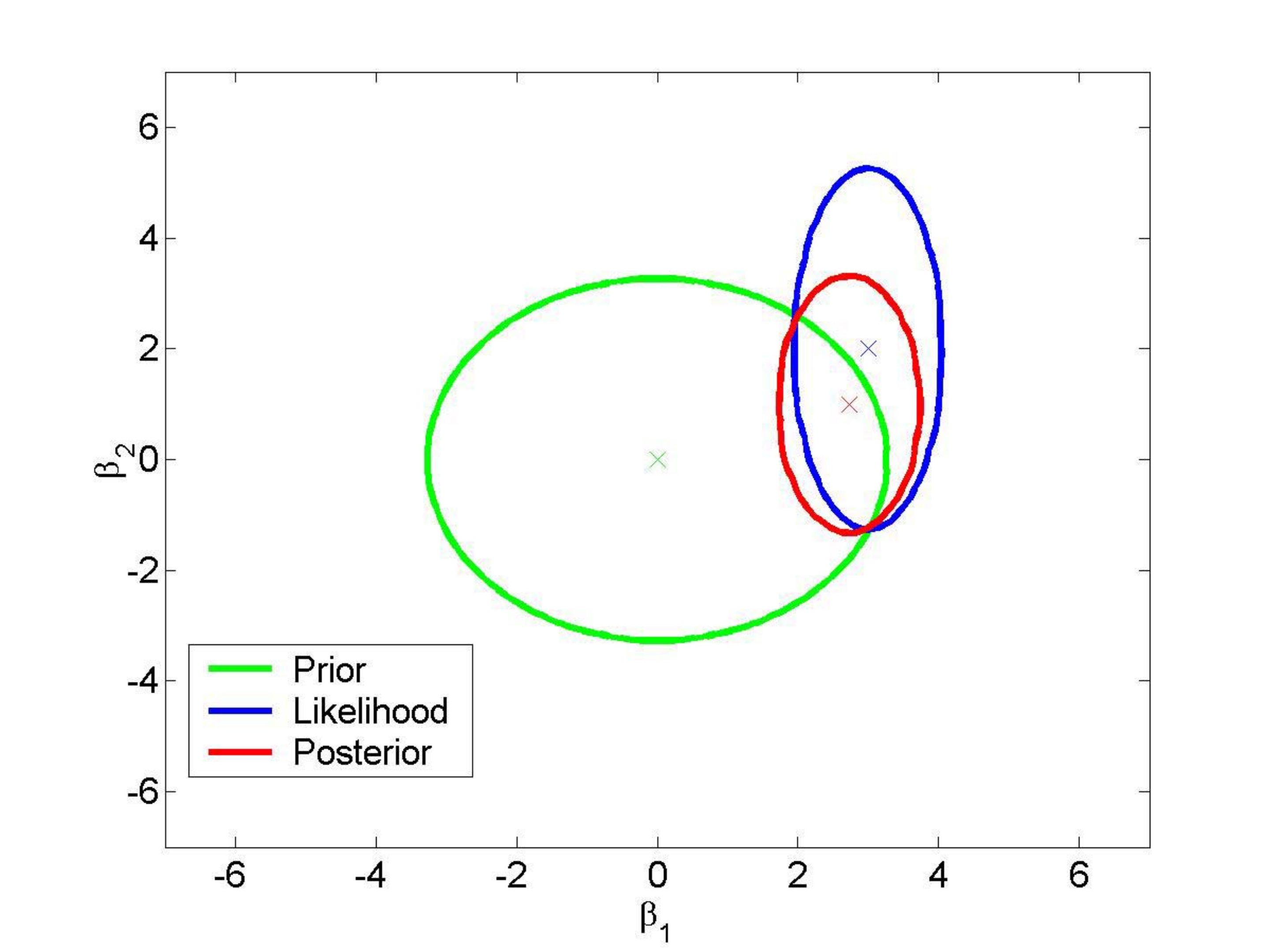
$$p(\theta^{(1)} \mid y) = N(m, P)$$

$$P = \lambda^{(1)} + \lambda^{(2)}$$

$$m = \frac{\lambda^{(1)}}{P}\theta^{(1)} + \frac{\lambda^{(2)}}{P}\theta^{(2)}$$

Relative Precision Weighting

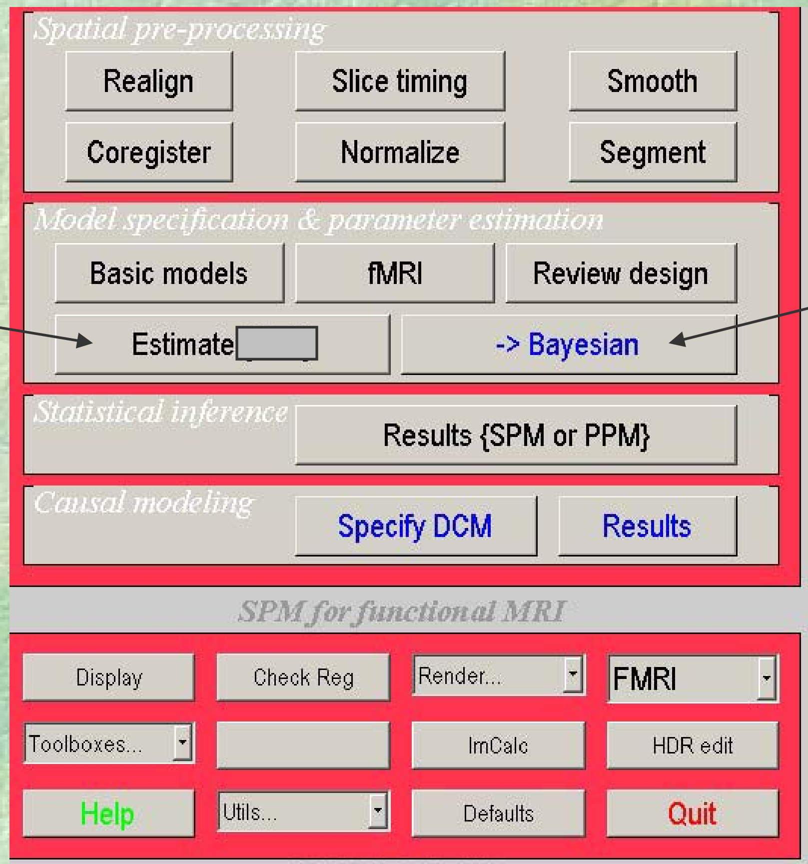
Two parameters



Posterior Probability Maps - PPMs

Least squares parameter estimation

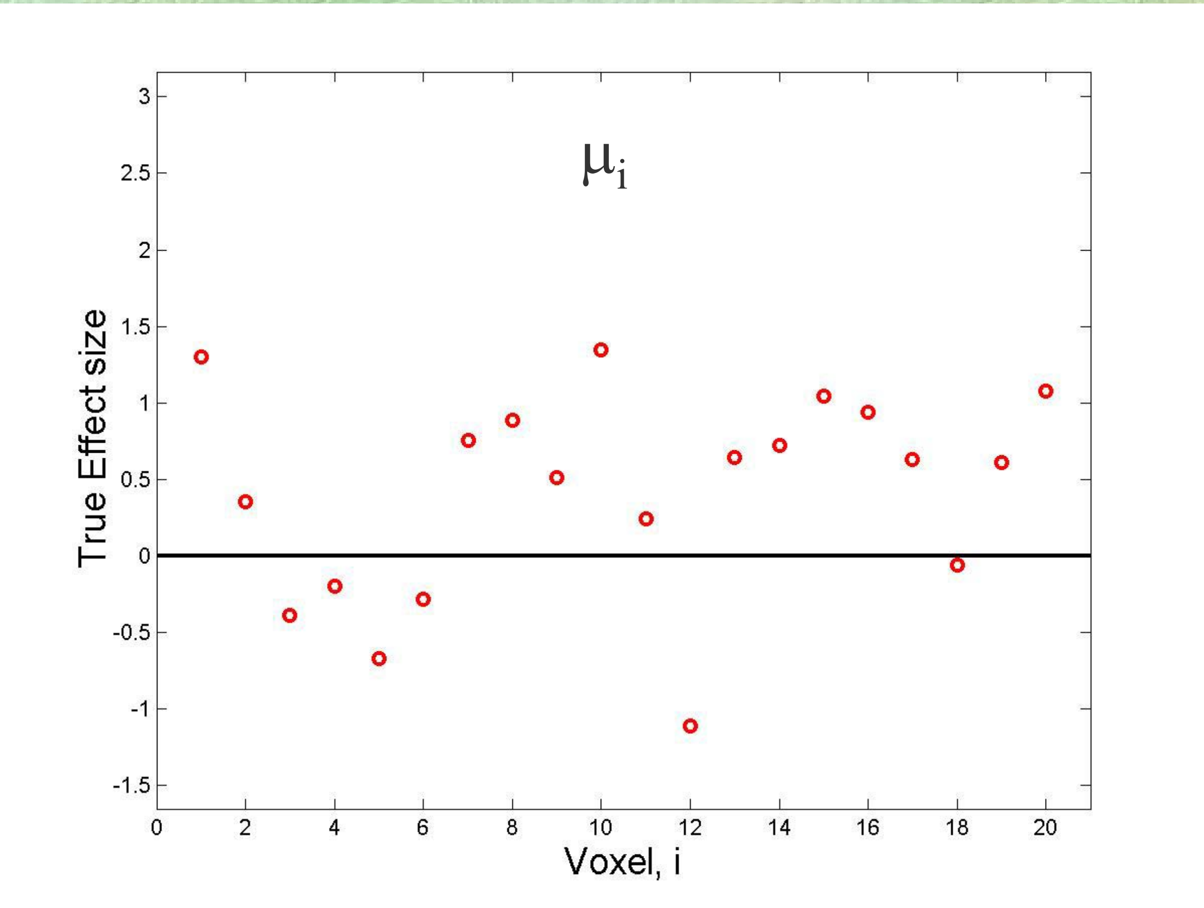
No Priors



Bayesian parameter estimation

Shrinkage priors

Prior

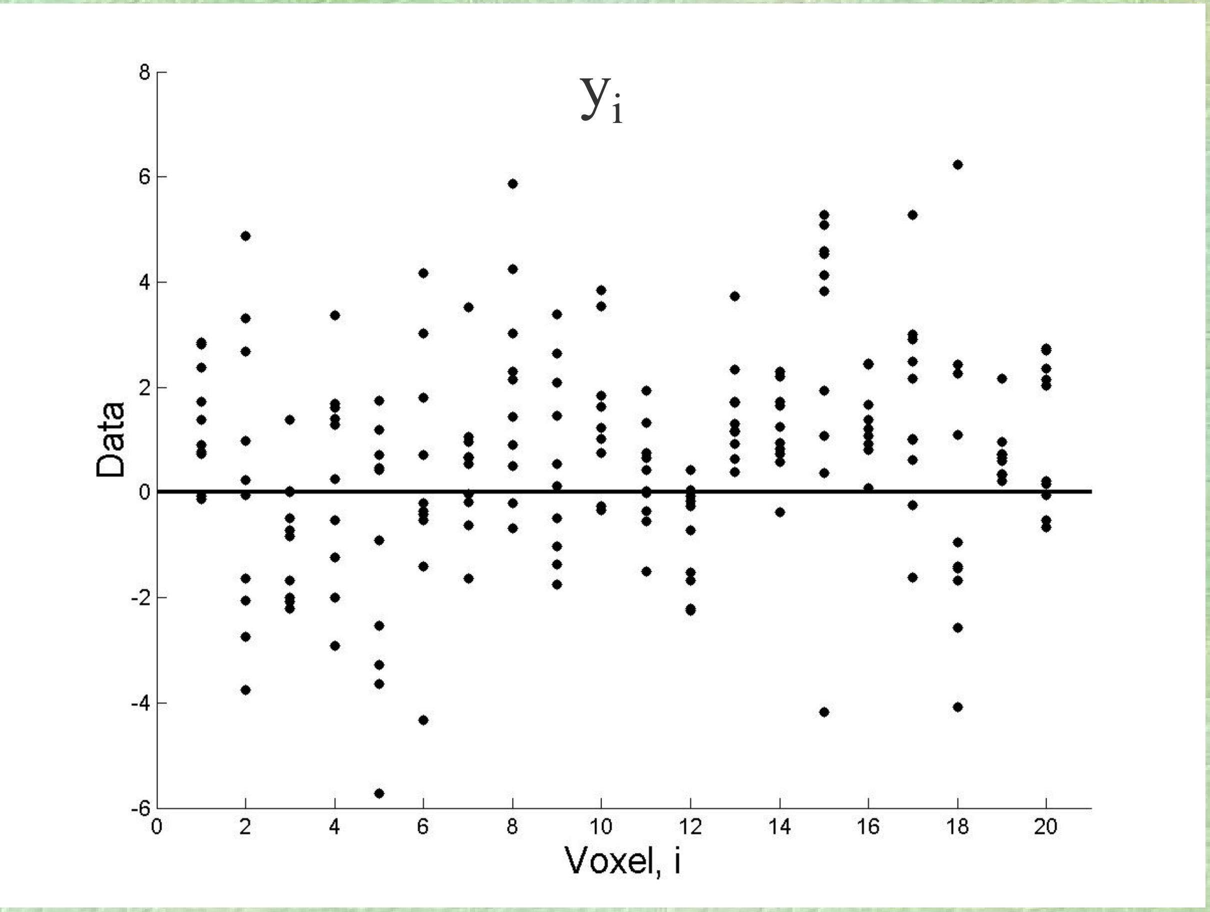


Prior:

$$p(\mu_i) = N(0, \alpha)$$

$1/\alpha=1$

Data



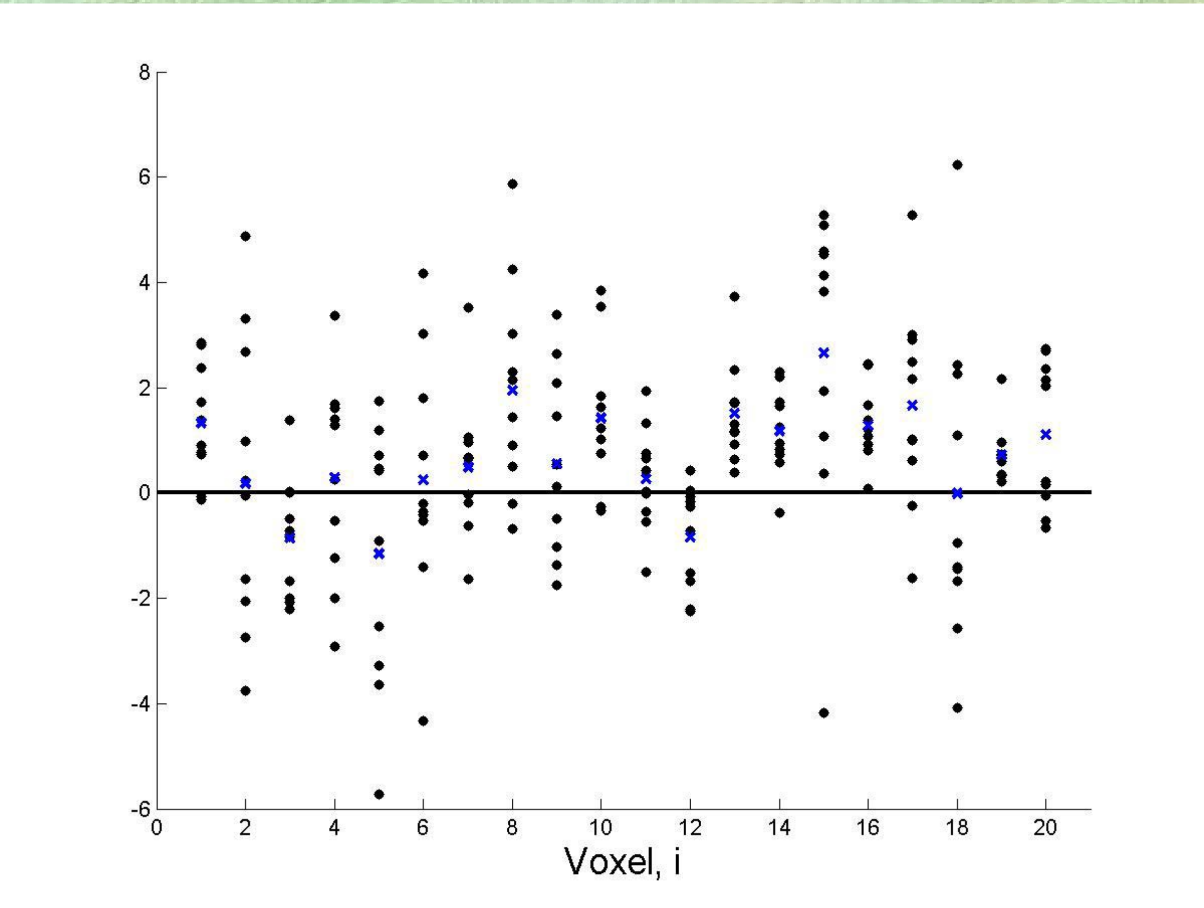
Likelihood:

$$p(y_{i,n} \mid \mu_i) = N(\mu_i, \beta_i \ )$$

i=1..V=20 voxels

n=1..N=10 scans

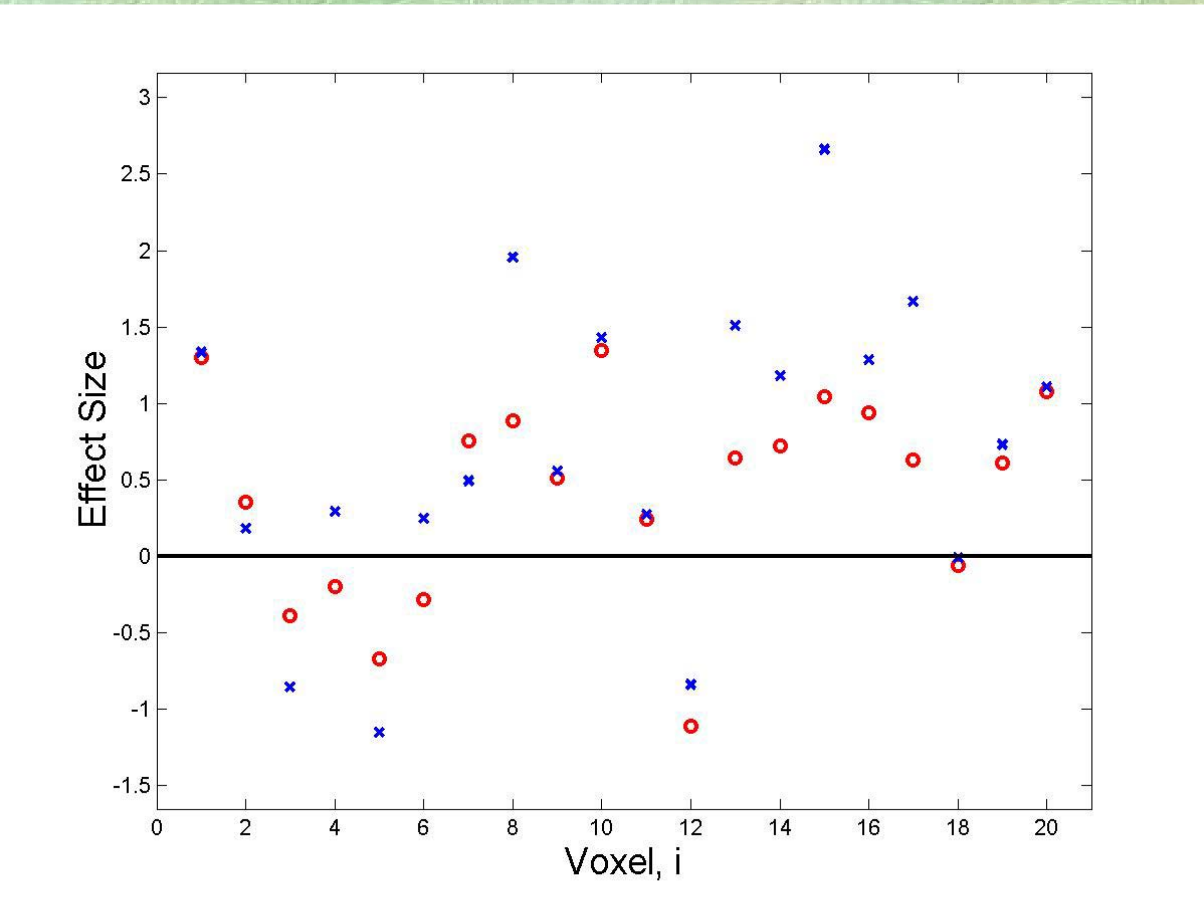
Least Squares Estimates



$$\hat{\mu}_i = \frac{1}{N}\sum_{n=1}^{N} y_{i,n}$$

$$\frac{1}{\widehat{\beta}_i} = \frac{1}{N-1}\sum_{n=1}^{N}\left(y_{i,n} - \hat{\mu}_i\right)^2$$

Least Squares Estimates

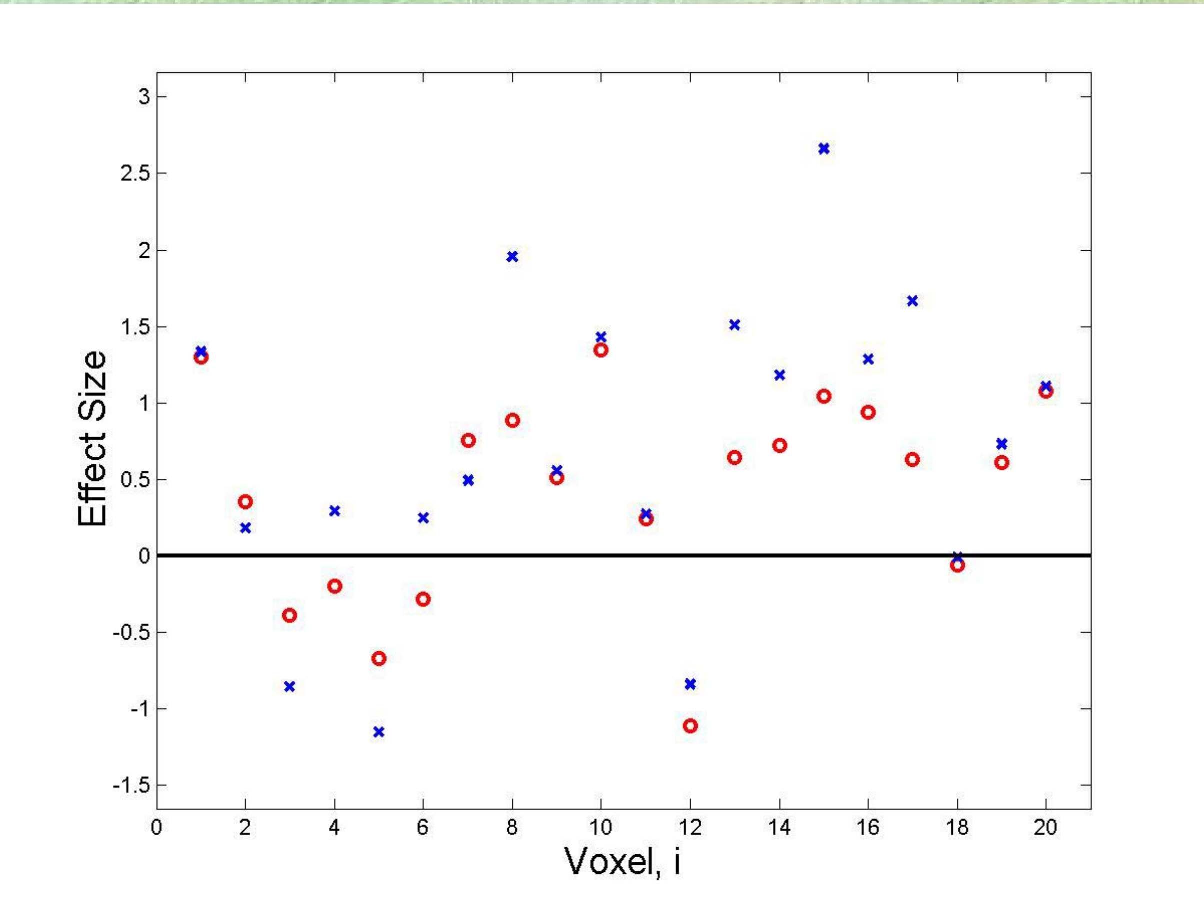


$$\hat{\mu}_i = \frac{1}{N}\sum_{n=1}^{N} y_{i,n}$$

$$\frac{1}{\hat{\beta}_i} = \frac{1}{N-1}\sum_{n=1}^{N}(y_{i,n} - \hat{\mu}_i)^2$$

Error=7.10

Bayesian Estimates



E-Step:

$$\hat{\mu}_i = \frac{\gamma_i}{N} \sum_{n=1}^{N} y_{i,n}$$

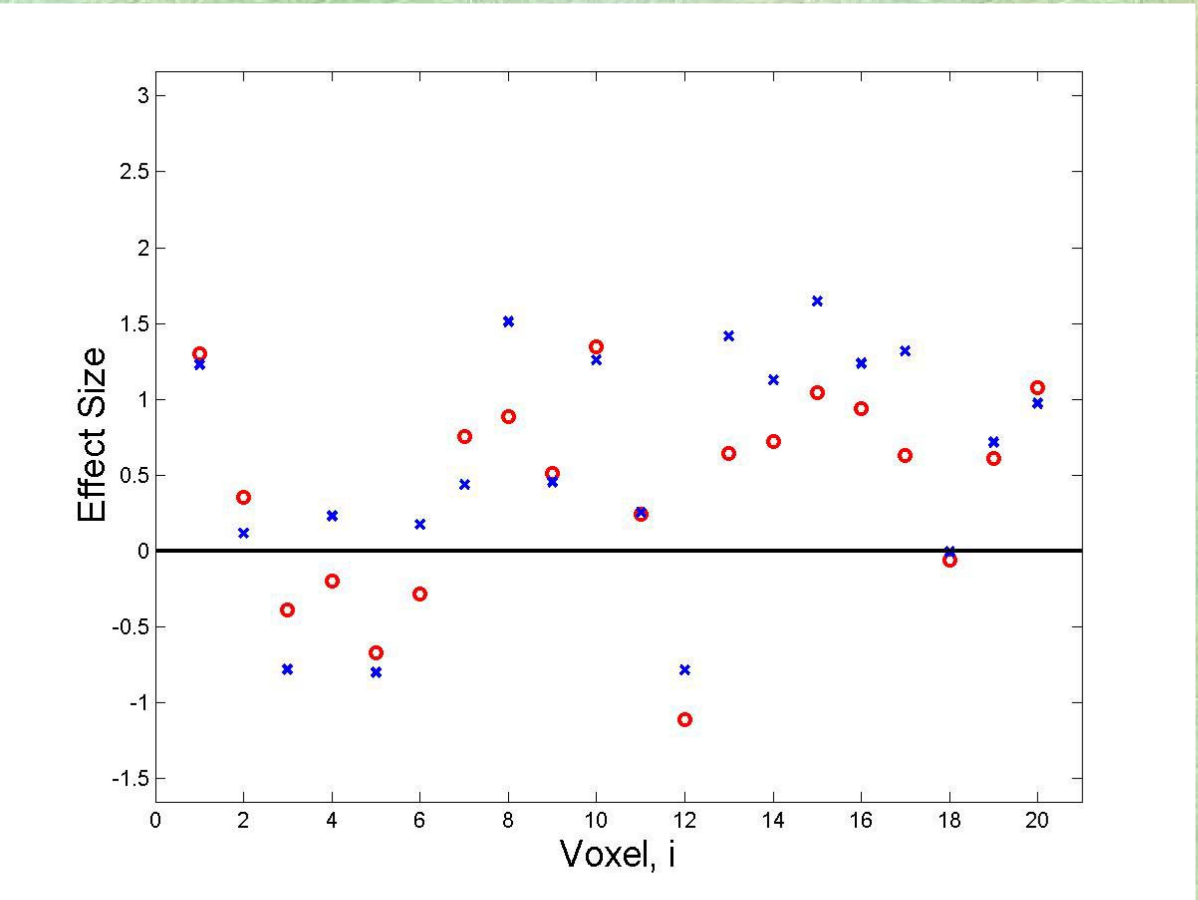
M-Step:

$$\frac{1}{\hat{\beta}_i} = \frac{1}{N - \gamma_i} \sum_{n=1}^{N} (y_{i,n} - \hat{\mu}_i)^2$$

$$\hat{\alpha} = 0 \qquad \gamma_i = 1$$

Iteration 1 Error=7.10

Bayesian Estimates



Iteration 2 Error=2.95

E-Step:

$$\hat{\mu}_i = \frac{\gamma_i}{N}\sum_{n=1}^{N} y_{i,n}$$

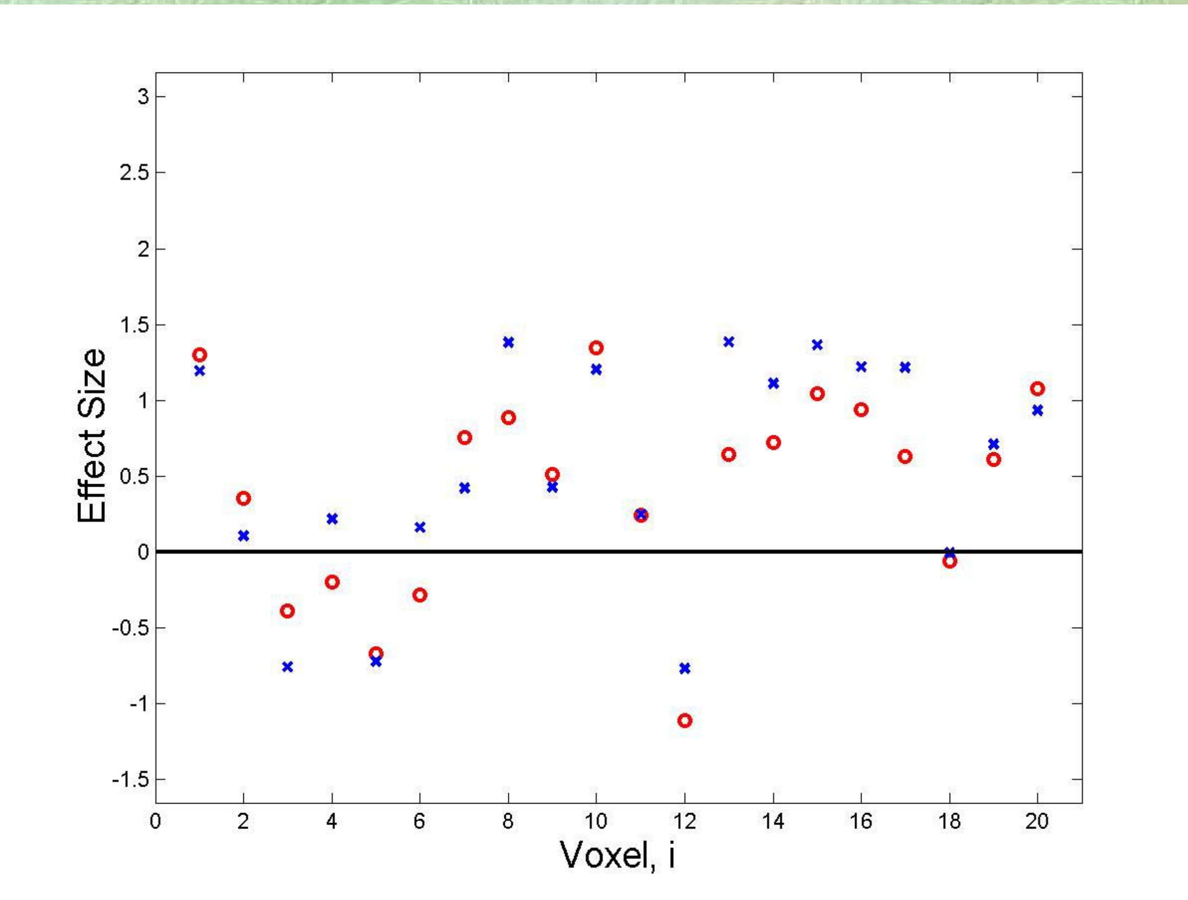
M-Step:

$$\frac{1}{\hat{\beta}_i} = \frac{1}{N-\gamma_i}\sum_{n=1}^{N}(y_{i,n} - \hat{\mu}_i)^2$$

$$\gamma_i = \frac{N\hat{\beta}_i}{N\hat{\beta}_i + \hat{\alpha}}$$

$$\frac{1}{\hat{\alpha}} = \frac{1}{\sum_i \gamma_i}\sum_{i=1}^{V}\hat{\mu}_i^{\,2} = 1.09$$

Bayesian Estimates



Iteration 3 Error=2.35

E-Step:

$$\hat{\mu}_i = \frac{\gamma_i}{N} \sum_{n=1}^{N} y_{i,n}$$

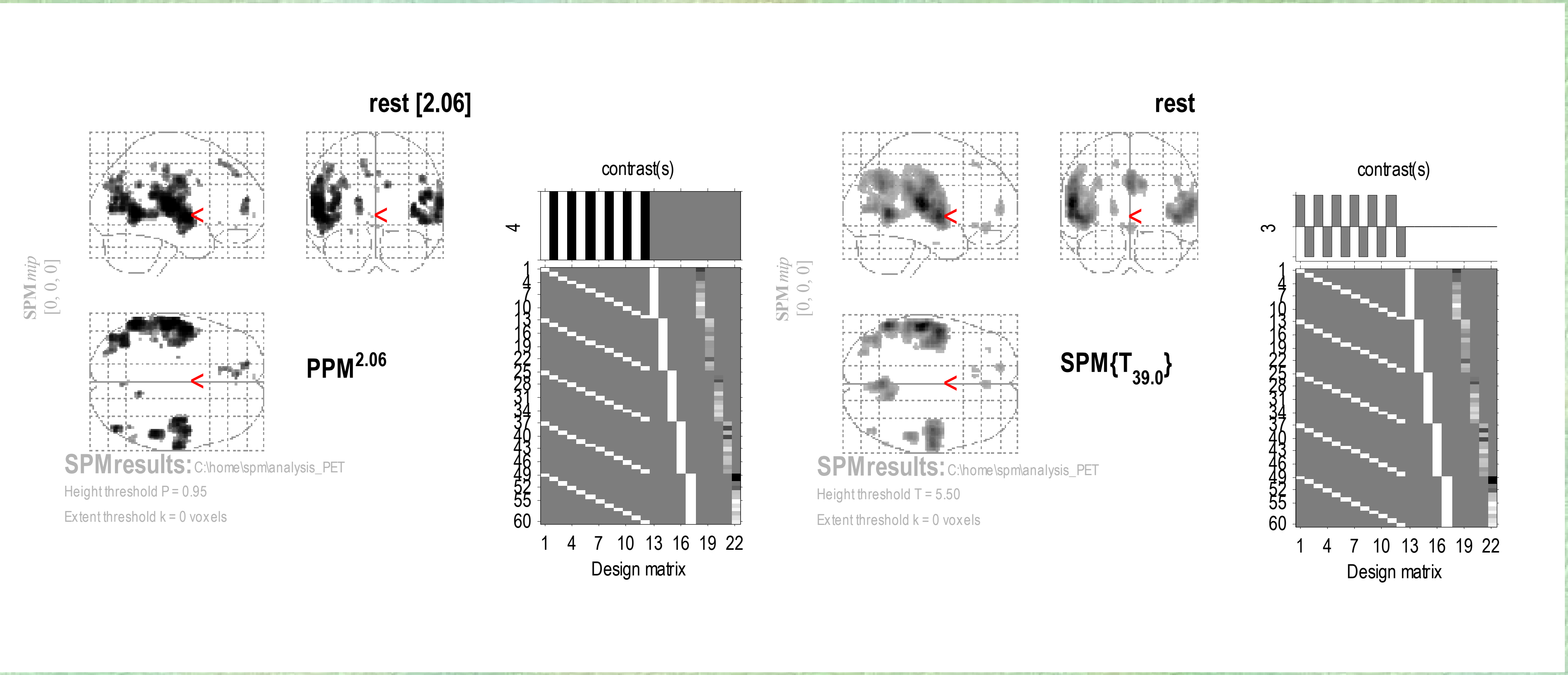
M-Step:

$$\frac{1}{\hat{\beta}_i} = \frac{1}{N - \gamma_i} \sum_{n=1}^{N} (y_{i,n} - \hat{\mu}_i)^2$$

$$\gamma_i = \frac{N \hat{\beta}_i}{N \hat{\beta}_i + \hat{\alpha}}$$

$$\frac{1}{\hat{\alpha}} = \frac{1}{\sum_i \gamma_i} \sum_{i=1}^{V} \hat{\mu}_i^{\,2} = 1.03$$

SPMs and PPMs

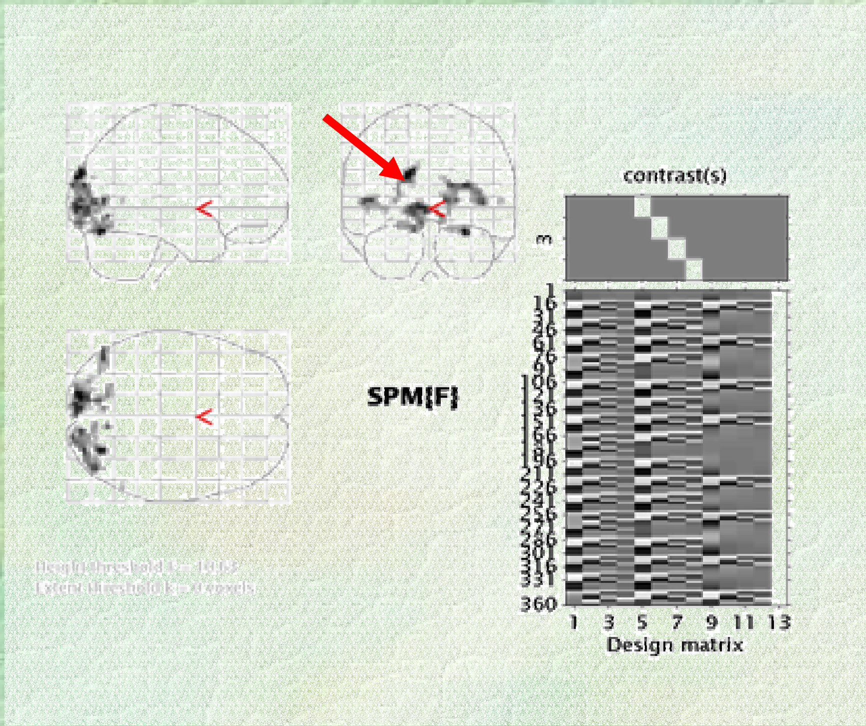


PPMs: Show activations of a given size

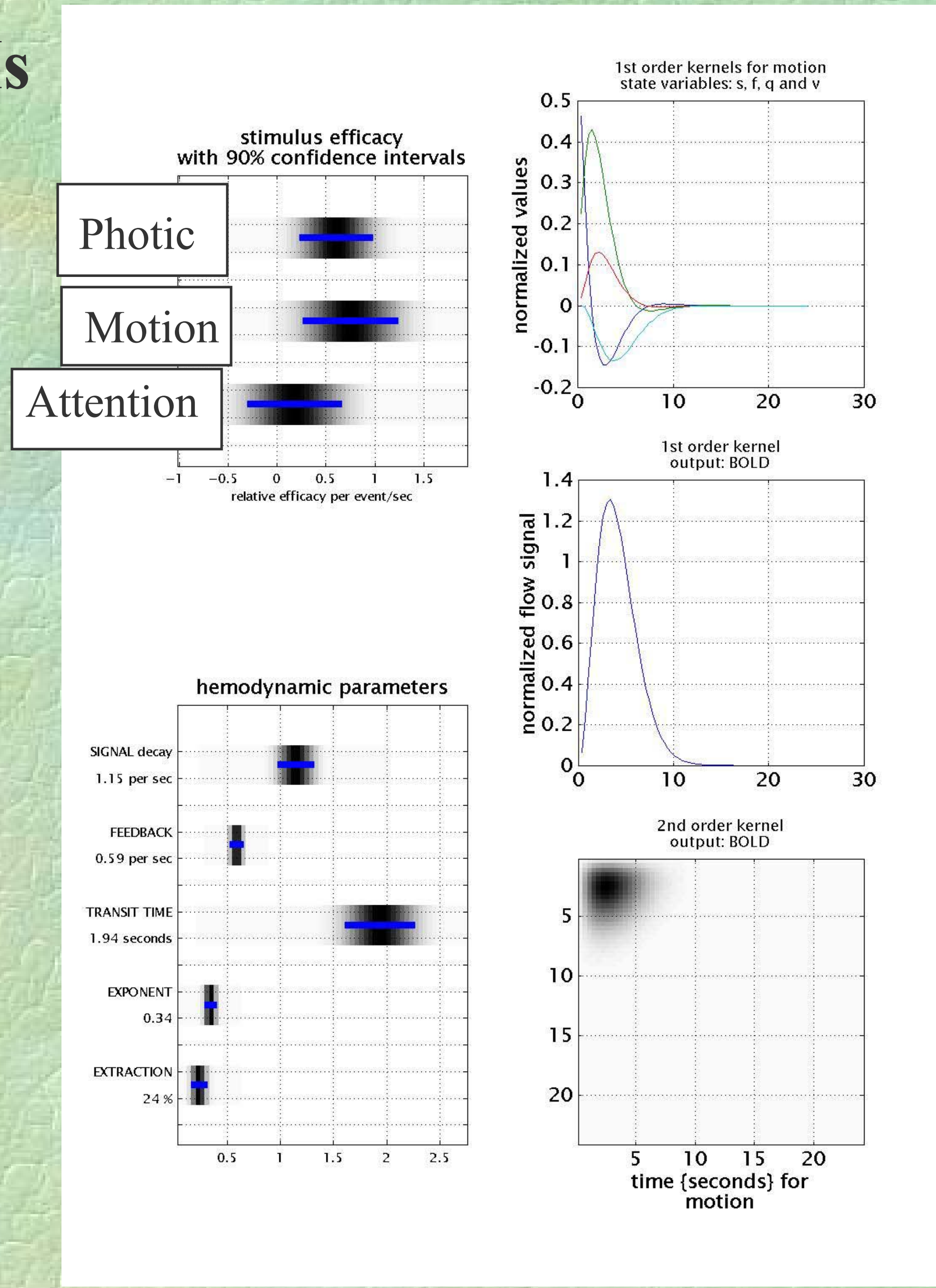
SPMs: show voxels with non-zero activations

Hemodynamic Models - HDMs

$$\dot{s} = \varepsilon_1 u_1(t) + \ldots + \varepsilon_J u_J(t) - \mathbf{s}/\tau_s - (f_{in} - 1)/\tau_f$$



fMRI study of attention to visual motion



PPMs

Advantages

One can infer a cause DID NOT elicit a response

SPMs conflate effect-size and effect-variability

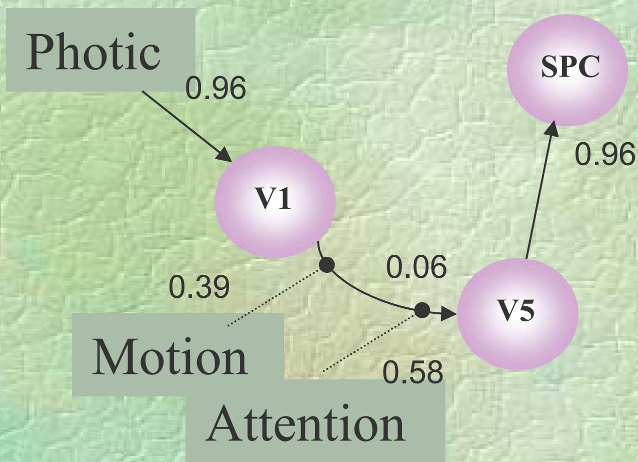
Disadvantages

Use of shrinkage priors over voxels is computationally demanding

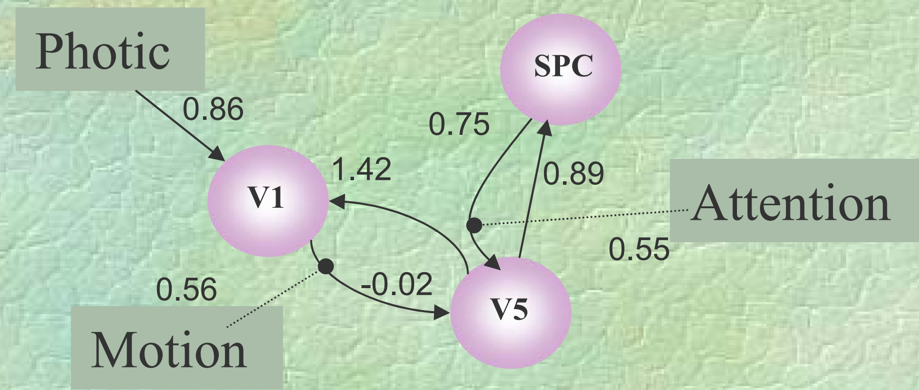
Utility of Bayesian approach is yet to be established

Comparing Dynamic Causal Models

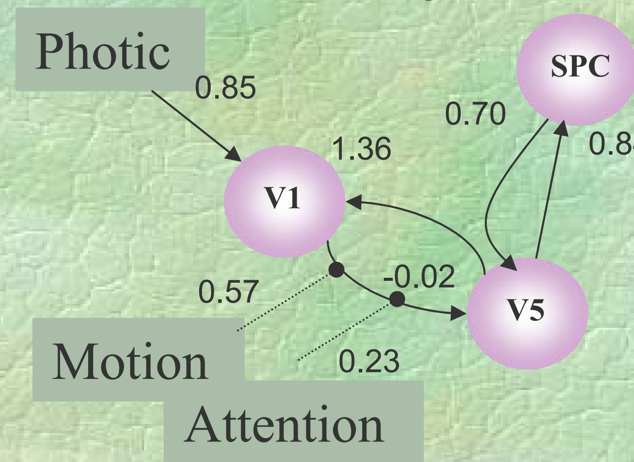
m=1



m=2



m=3



Bayesian Evidence:

$$p(y \mid m) = \int p(y \mid m, \theta) p(\theta \mid m) d\theta$$

Bayes factors:

$$B_{12} = \frac{p(y \mid m=1)}{p(y \mid m=2)}$$