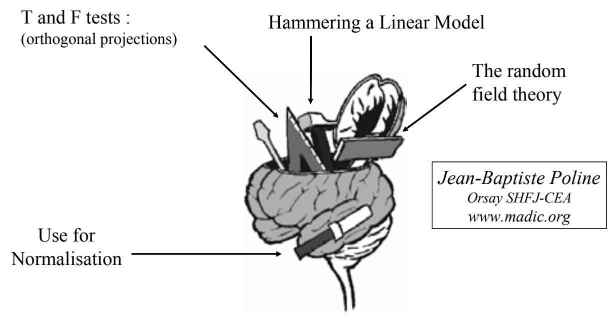
SPM short course at Yale – April 2005
Linear Models and Contrasts



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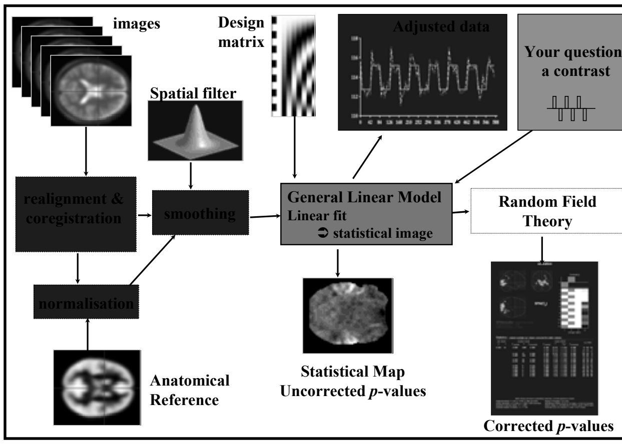
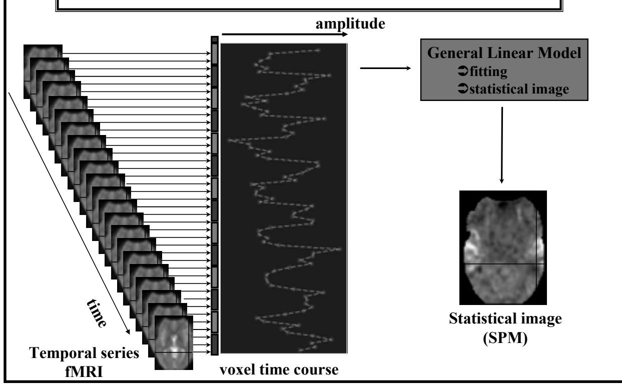


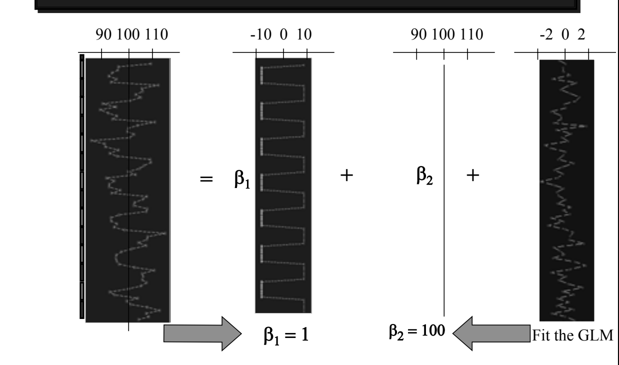
Plan

- *Make sure we know all about the estimation (fitting) part ....*
- *Make sure we understand the testing procedures : t- and F-tests*
- *A bad model ... And a better one*
- *Correlation in our model : do we mind ?*
- *A (nearly) real example*

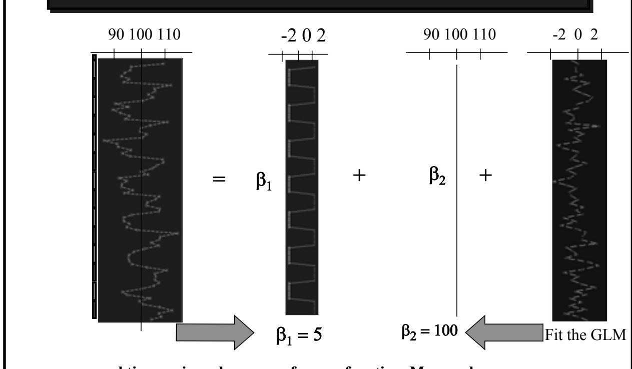
One voxel = One test (t, F, ...)



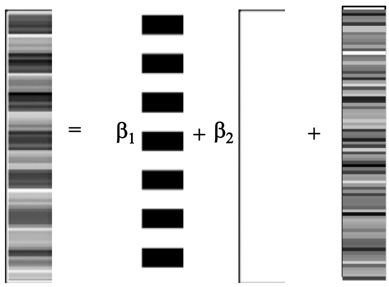
Regression example…



Regression example…

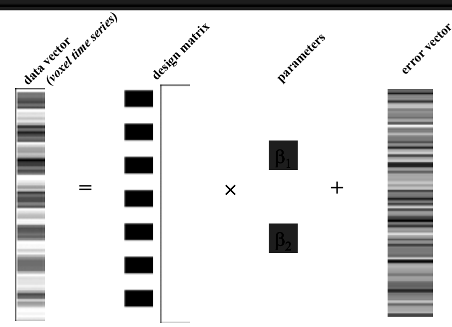


…revisited : matrix form



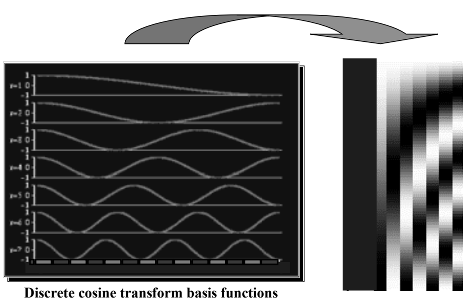
$$Y = \beta_1 \times f(t) + \beta_2 \times 1 + \varepsilon_s$$

Box car regression: design matrix…



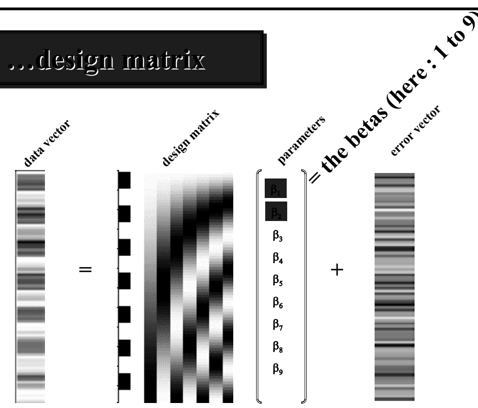
$$\underline{Y} = X \times \underline{\beta} + \underline{\varepsilon}$$

Add more reference functions ...



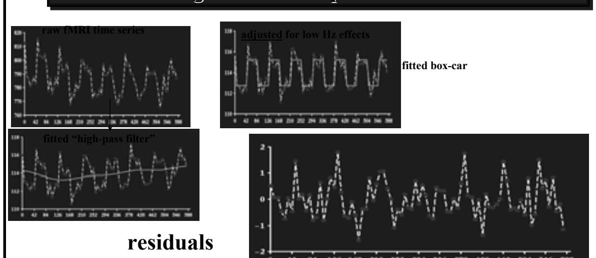
Discrete cosine transform basis functions

…design matrix



$$Y = X \times \beta + \varepsilon$$

Fitting the model = finding some estimate of the betas = minimising the sum of square of the residuals S^2



$$\frac{\sum \text{the squared values of the residuals}}{\text{number of time points minus the number of estimated betas}} = s^2$$

Summary ...

- *We put in our model regressors (or covariates) that represent how we think the signal is varying (of interest and of no interest alike)*
- *Coefficients (= parameters) are estimated using the Ordinary Least Squares (OLS) or Maximum Likelihood (ML) estimator.*
- *These estimated parameters (the "betas")* ***depend*** *on the scaling of the regressors. But entered with SPM, regressors are normalised and comparable.*
- *The residuals, their sum of squares and the resulting tests (t,F),* ***do not*** *depend on the scaling of the regressors.*

Plan

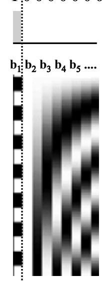
- *Make sure we all know about the estimation (fitting) part ....*
- *Make sure we understand t and F tests*
- *A (nearly) real example*
- *A bad model ... And a better one*
- *Correlation in our model : do we mind ?*

T test - one dimensional contrasts - SPM{*t*}

A contrast **= a linear combination of parameters: $c' \times \beta$**

c' = 1 0 0 0 0 0 0 0

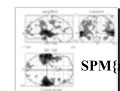
b_1 b_2 b_3 b_4 b_5 ....



box-car amplitude > 0 ?
=
$\beta_1 > 0$?
=>

Compute $1 \times b_1 + 0 \times b_2 + 0 \times b_3 + 0 \times b_4 + 0 \times b_5 + \ldots$
and
divide by estimated standard deviation

$$T = \frac{\textit{contrast}\text{ of estimated parameters}}{\sqrt{\text{variance estimate}}} \qquad T = \frac{c'b}{\sqrt{s^2 c'(X'X)^+ c}}$$



How is this computed ? (t-test)

$\frac{\textit{contrast}\text{ of estimated parameters}}{\sqrt{\text{variance estimate}}}$

Estimation [Y, X] [b, s]

$Y = X\beta + \varepsilon$ $\qquad \varepsilon \sim \sigma^2 N(0,I)$ *(Y : at one position)*

$b = (X'X)^+ X'Y$ $\qquad$ *(b: estimate of β) -> beta??? images*

$e = Y - Xb$ $\qquad$ *(e: estimate of ε)*

$s^2 = (e'e/(n - p))$ $\qquad$ *(s: estimate of σ, n: time points, p: parameters)* *-> 1 image ResMS*

Test [b, s^2, c] [c'b, t]

$\mathrm{Var}(c'b) = s^2 c'(X'X)^+ c$ $\qquad$ *(compute for each contrast c)*

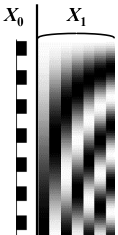
$t = c'b / \mathrm{sqrt}(s^2 c'(X'X)^+ c)$ $\qquad$ *(c'b -> images spm_con???*
compute the t images -> images spm_t???)

under the null hypothesis H_0 : $t \sim$ Student-t(df) $\qquad$ *df = n-p*

F-test (SPM{*F*}) : a reduced model or ...

Tests multiple linear hypotheses : Does X1 model anything ?

H_0: True (reduced) model is X_0



This (full) model ?

Or this one?

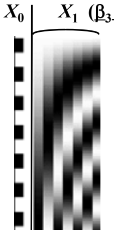
$$F = \frac{\text{additional variance accounted for by tested effects}}{\text{error variance estimate}}$$

$F \sim (S_0^2 - S^2) / S^2$

F-test (SPM{*F*}) : a reduced model or ... multi-dimensional contrasts ?

tests multiple linear hypotheses. Ex : does DCT set model anything?

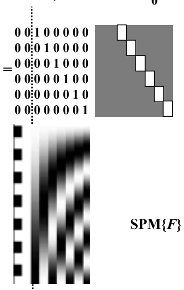
H_0: True model is X_0 $\qquad$ **H_0: $\beta_{3\text{-}9}$ = (0 0 0 0 ...)** $\qquad$ **test H_0 : $c' \times b = 0$?**

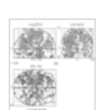


This model ?

Or this one ?

c' =
0 0 1 0 0 0 0 0
0 0 0 1 0 0 0 0
0 0 0 0 1 0 0 0
0 0 0 0 0 1 0 0
0 0 0 0 0 0 1 0
0 0 0 0 0 0 0 1





How is this computed ? (F-test)

$\frac{\text{additional variance accounted for by tested effects}}{\text{Error variance estimate}}$

Estimation [Y, X] [b, s]

$Y = X\beta + \varepsilon$ $\qquad \varepsilon \sim N(0, \sigma^2 I)$

$Y = X_0\beta_0 + \varepsilon_0$ $\qquad \varepsilon_0 \sim N(0, \sigma_0^2 I)$ $\qquad$ *X_0 : X Reduced*

Estimation [Y, X_0] [b_0, s_0]

$b_0 = (X_0'X_0)^+ X_0'Y$

$e_0 = Y - X_0 b_0$ $\qquad$ *(e_0: estimate of ε_0)*

$s^2_0 = (e_0'e_0/(n - p_0))$ $\qquad$ *(s_0: estimate of σ_0, n: time, p_0: parameters)*

Test [b, s, c] [ess, F]

$F \sim (s_0 - s) / s^2$ $\qquad$ *-> image spm_ess???*
-> image of F : spm_F???

under the null hypothesis : $F \sim F(p - p0, n-p)$

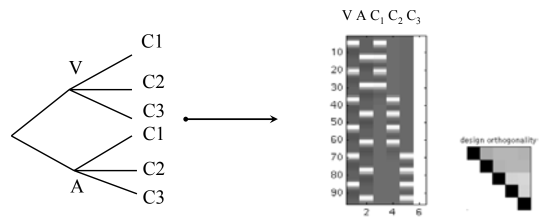
Plan

- *Make sure we all know about the estimation (fitting) part ....*
- *Make sure we understand t and F tests*
- *A (nearly) real example : testing main effects and interactions*
- *A bad model ... And a better one*
- *Correlation in our model : do we mind ?*

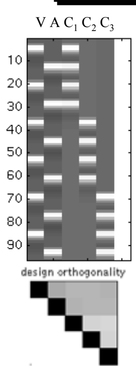
A real example (almost !)

Experimental Design ⟶ Design Matrix

Factorial design with 2 factors : modality and category
2 levels for modality (eg Visual/Auditory)
3 levels for category (eg 3 categories of words)



Asking ourselves some questions ...



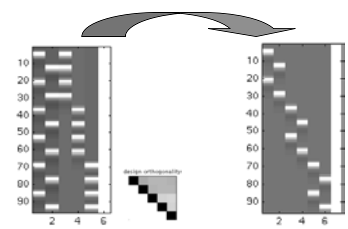
Test C1 > C2 : c = [0 0 1 -1 0 0]
Test V > A : c = [1 -1 0 0 0 0]

Test C1,C2,C3 ? (F) c = [0 0 1 0 0 0] [0 0 0 1 0 0] [0 0 0 0 1 0]

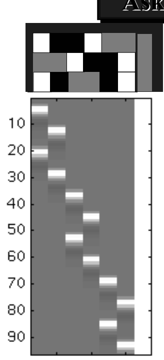
Test the interaction MxC ?

- Design Matrix not orthogonal
- Many contrasts are non estimable
- Interactions MxC are not modelled

Modelling the interactions



Asking ourselves some questions ...



Test C1 > C2 : c = [1 1 -1 -1 0 0 0]

Test V > A : c = [1 -1 1 -1 1 -1 0]

Test the categories :

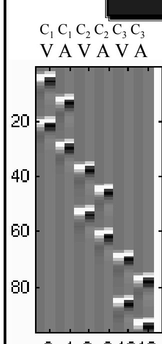
c = [1 1 -1 -1 0 0 0] [0 0 1 1 -1 -1 0] [1 1 0 0 -1 -1 0]

Test the interaction MxC :

c = [1 -1 -1 1 0 0 0] [0 0 1 -1 -1 1 0] [1 -1 0 0 -1 1 0]

- Design Matrix orthogonal
- All contrasts are estimable
- Interactions MxC modelled
- If no interaction ... ? Model is too “big” !

Asking ourselves some questions ... With a more flexible model



Test C1 > C2 ?
Test C1 different from C2 ?
from

c = [1 1 -1 -1 0 0 0]

to

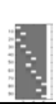
c = [1 0 1 0 -1 0 -1 0 0 0 0 0 0] [0 1 0 1 0 -1 0 -1 0 0 0 0 0]

becomes an F test!

Test V > A ?

c = [1 0 -1 0 1 0 -1 0 1 0 -1 0 0]

is possible, but is OK only if the regressors coding for the delay are all equal



Plan

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- *Make sure we understand t and F tests*
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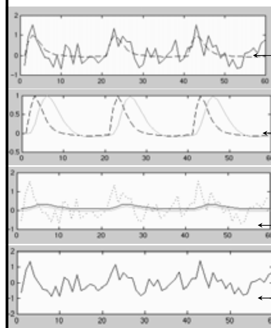
- *A bad model ... And a better one*
- *Correlation in our model : do we mind ?*

A bad model ...



True signal and observed signal (---)

Model (green, pic at 6sec)
TRUE signal (blue, pic at 3sec)

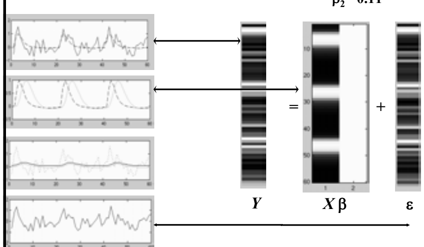
Fitting (b1 = 0.2, mean = 0.11)

Residual (still contains some signal)

=> Test for the green regressor not significant

A bad model ...

β_1= 0.22
β_2= 0.11

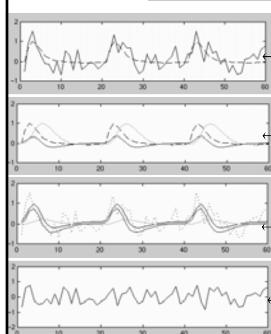


Residual Variance = 0.3

$P(Y|\ \beta_1 = 0)$ =>
p-value = 0.1
(t-test)

$P(Y|\ \beta_1 = 0)$ =>
p-value = 0.2
(F-test)

A « better » model ...



True signal + observed signal

Model (green and red)
and true signal (blue ---)
Red regressor : temporal derivative of the green regressor

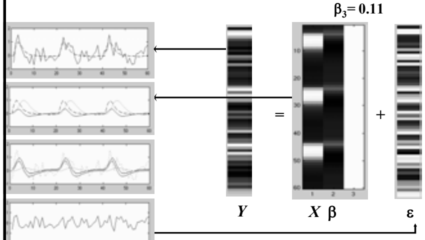
Global fit (blue)
and partial fit (green & red)
Adjusted and fitted signal

Residual (a smaller variance)

=> t-test of the green regressor significant
=> F-test very significant
=> t-test of the red regressor very significant

A better model ...

β_1= 0.22
β_2= 2.15
β_3= 0.11

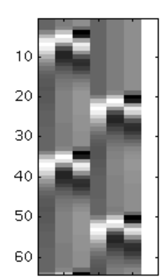


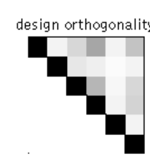
Residual Var = 0.2

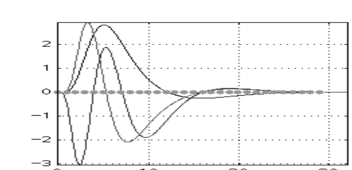
$P(Y|\ \beta_1 = 0)$
p-value = 0.07
(t-test)

$P(Y|\ \beta_1 = 0, \beta_2 = 0)$
p-value = 0.000001
(F-test)

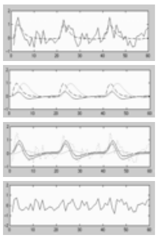
Flexible models : Gamma Basis







Summary ... (2)



- *The residuals should be looked at ...!*
- *Test flexible models if there is little a priori information*
- *In general, use the F-tests to look for an overall effect, then look at the response shape*

- *Interpreting the test on a single parameter (one regressor) can be difficult: cf the delay or magnitude situation*
- *BRING ALL PARAMETERS AT THE 2nd LEVEL*

Plan

- *Make sure we all know about the estimation (fitting) part ....*
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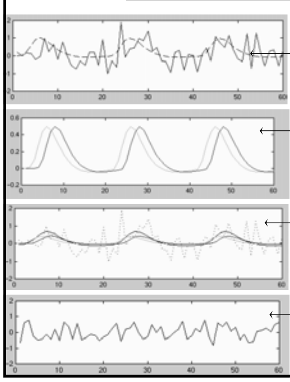
- *Correlation in our model : do we mind ?*

?

Correlation between regressors



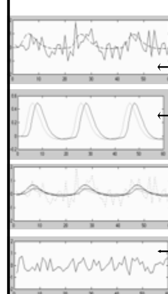
True signal

Model (green and red)

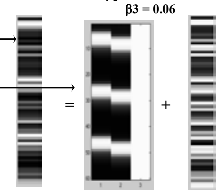
Fit (blue : global fit)

Residual

Correlation between regressors



$\beta_1 = 0.79$
$\beta_2 = 0.85$
$\beta 3 = 0.06$



Y = $X\beta$ + ε

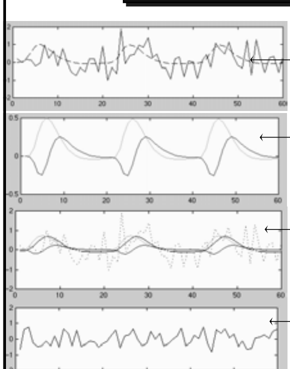
Residual var. = 0.3

$P(Y|\ \beta_1 = 0)$
p-value = 0.08
(t-test)

$P(Y|\ \beta_2 = 0)$
p-value = 0.07
(t-test)

$P(Y|\ \beta_1 = 0, \beta_2 = 0)$
p-value = 0.002
(F-test)

Correlation between regressors - 2



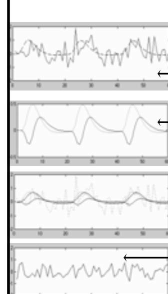
true signal

Model (green and red)
red regressor has been orthogonalised with respect to the green one
⇔ remove everything that correlates with the green regressor

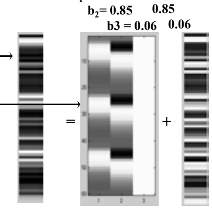
Fit

Residual

Correlation between regressors -2



$b_1 = 1.47$ 0.79***
$b_2 = 0.85$ 0.85
$b3 = 0.06$ 0.06



Y = $X\beta$ + ε

Residual var. = 0.3

$P(Y|\ \beta_1 = 0)$
p-value = 0.0003
(t-test)

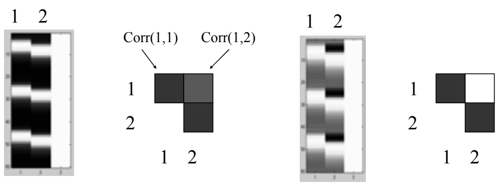
$P(Y|\ \beta_2 = 0)$
p-value = 0.07
(t-test)

$P(Y|\ \beta_1 = 0, \beta_2 = 0)$
p-value = 0.002
(F-test)

See « explore design »

Design orthogonality : « explore design »

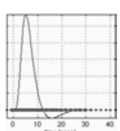
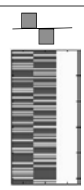
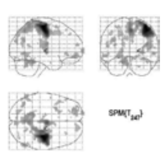
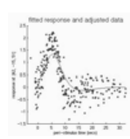
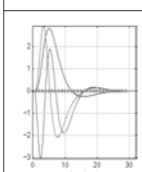
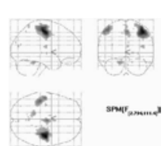
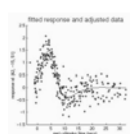
Black = completely correlated White = completely orthogonal



Beware: when there are more than 2 regressors (C1,C2,C3,...), you may think that there is little correlation (light grey) between them, but C1 + C2 + C3 may be correlated with C4 + C5

Summary ... (3)

- *We implicitly test for an* *__additional__* *effect only, be careful if there is correlation*
- *Orthogonalisation = decorrelation*
  - This is not generally needed
  - Parameters and test on the non modified regressor change
- *It is always simpler to have orthogonal regressors and therefore designs !*
- *In case of correlation, use F-tests to see the overall significance. There is generally no way to decide to which regressor the « common » part should be attributed to*

Convolution model	Design and contrast	SPM(t) or SPM(F)	Fitted and adjusted data
			
			

Conclusion : check your models

- *Check your residuals/model*
  - multivariate toolbox
- *Check your HRF form*
  - HRF toolbox
- *Check group homogeneity*
  - Distance toolbox

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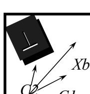
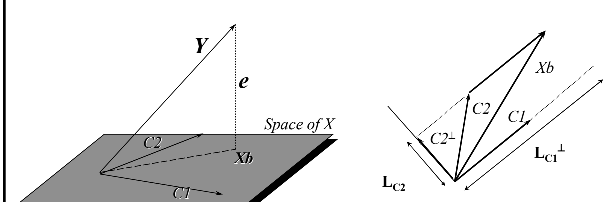


Implicit or explicit (⊥) decorrelation (or orthogonalisation)



This generalises when testing several regressors (F tests)

cf Andrade et al., NeuroImage, 1999

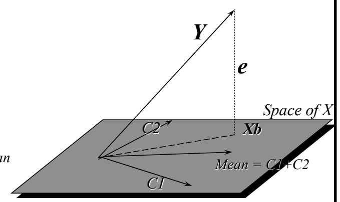
L_{C2} : test of C2 in the implicit ⊥ model

$L_{C1}^{\perp}$: test of C1 in the explicit ⊥ model

“completely” correlated ...

$$Y = Xb + e; \quad X = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$$

Cond 1 Cond 2 Mean



Parameters are not unique in general ! Some contrasts have no meaning: NON ESTIMABLE

c = [1 0 0] is **not** estimable (no specific information in the first regressor);

c = [1 -1 0] is estimable;