Vancouver course – August 2010
Linear Models – Contrasts

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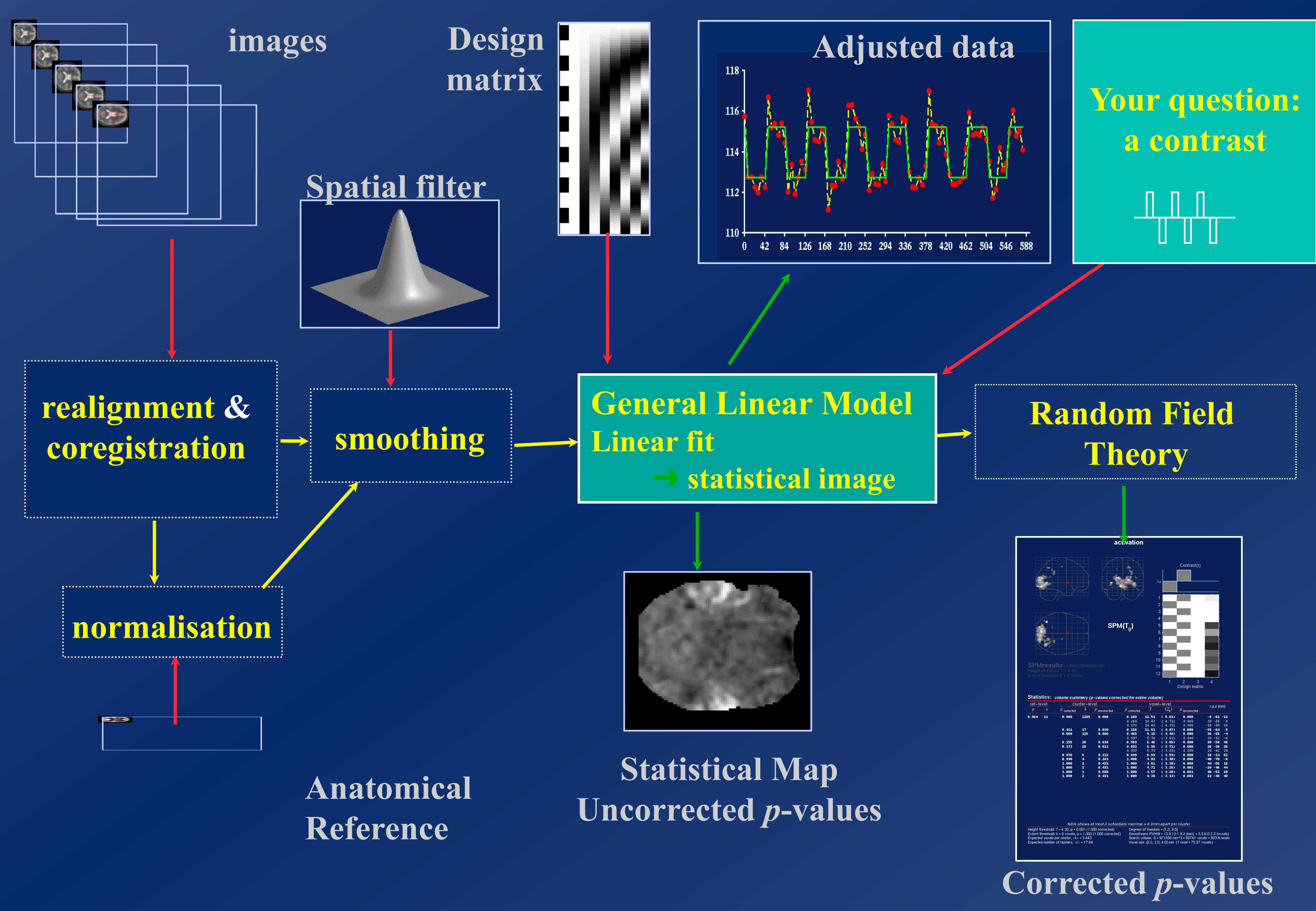
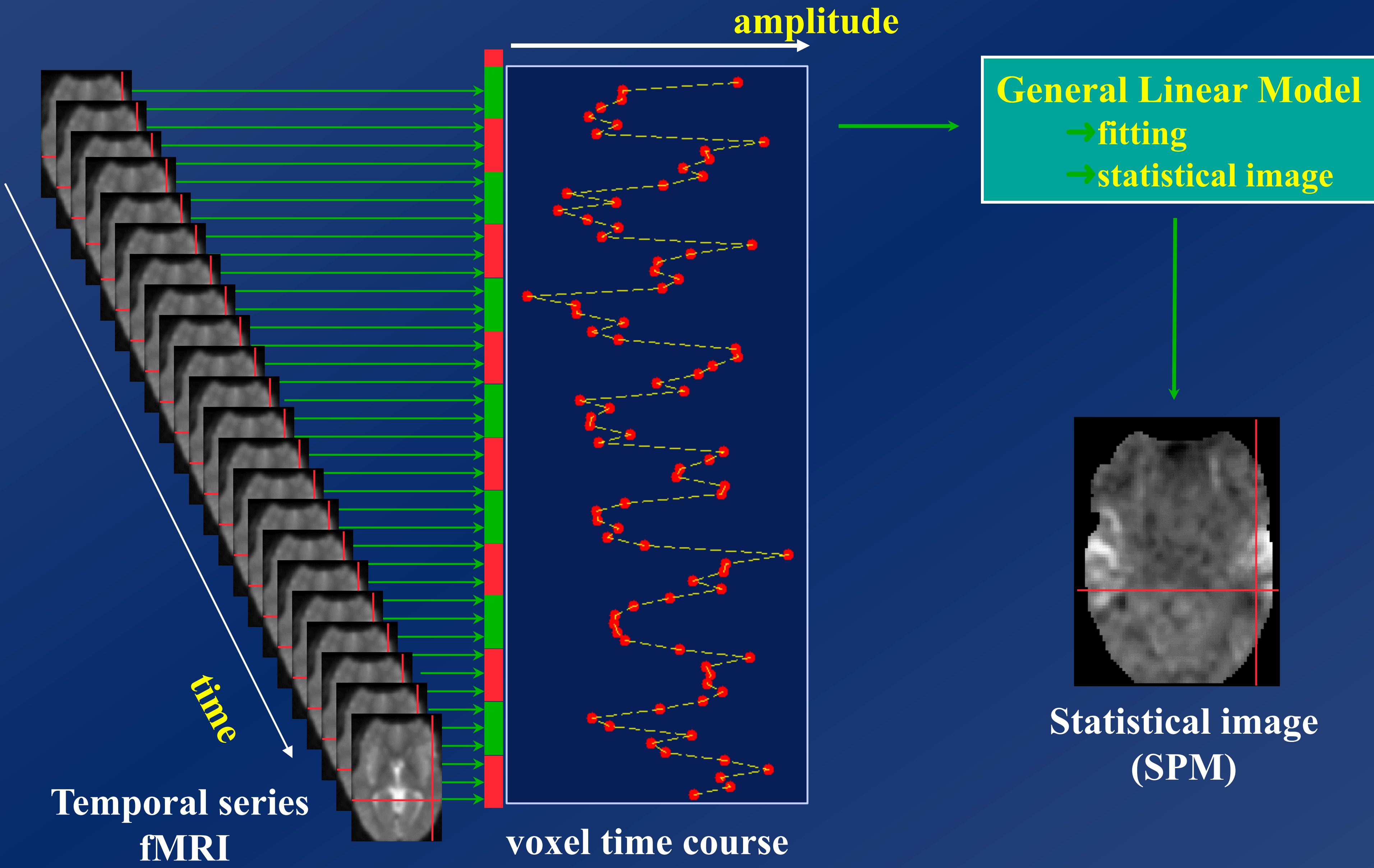


images
Design matrix
Adjusted data
118
116
114
112
110
0
42
84
126
168
210
252
294
336
378
420
462
504
546
588
Your question: a contrast
Spatial filter
realignment & coregistration
smoothing
General Linear Model
Linear fit
→ statistical image
Random Field Theory
normalisation
Anatomical Reference
Statistical Map
Uncorrected p-values
Corrected p-values

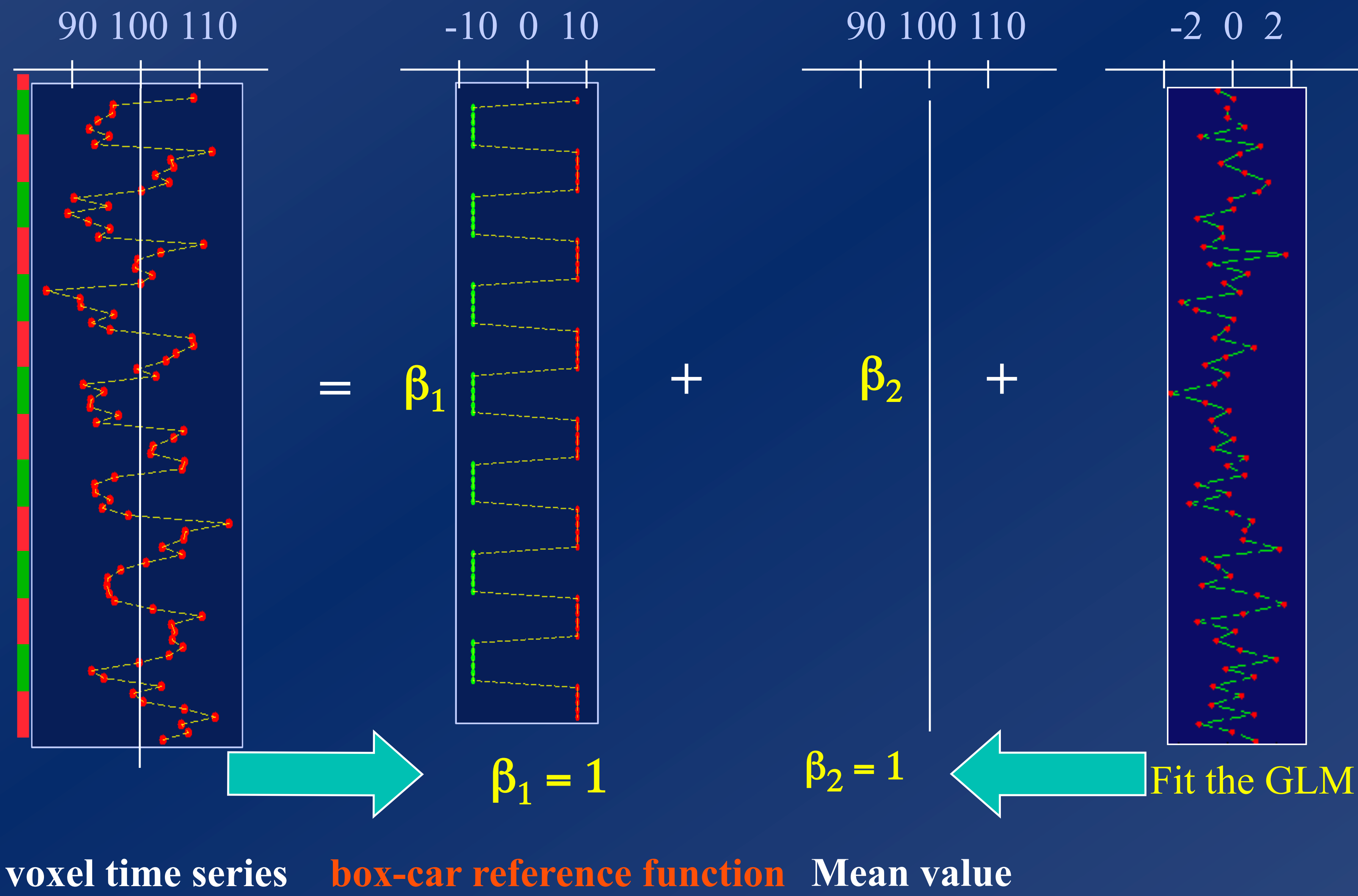
Plan

- *REPEAT: model and fitting the data with a Linear Model*
- *Make sure we understand the testing procedures : t- and F-tests*
- *But what do we test exactly ?*
- *Examples – almost real*

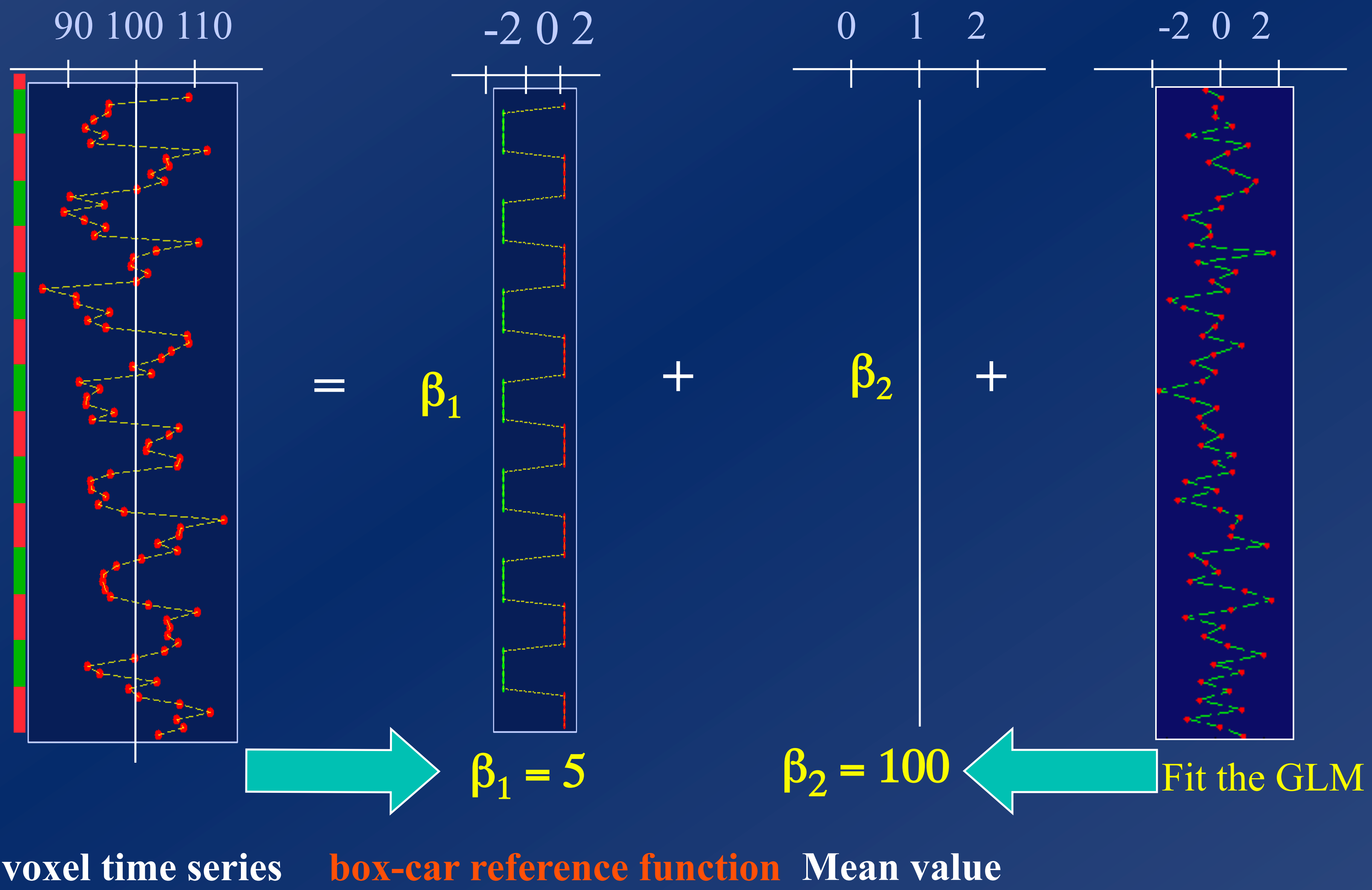
One voxel = One test (t, F, ...)



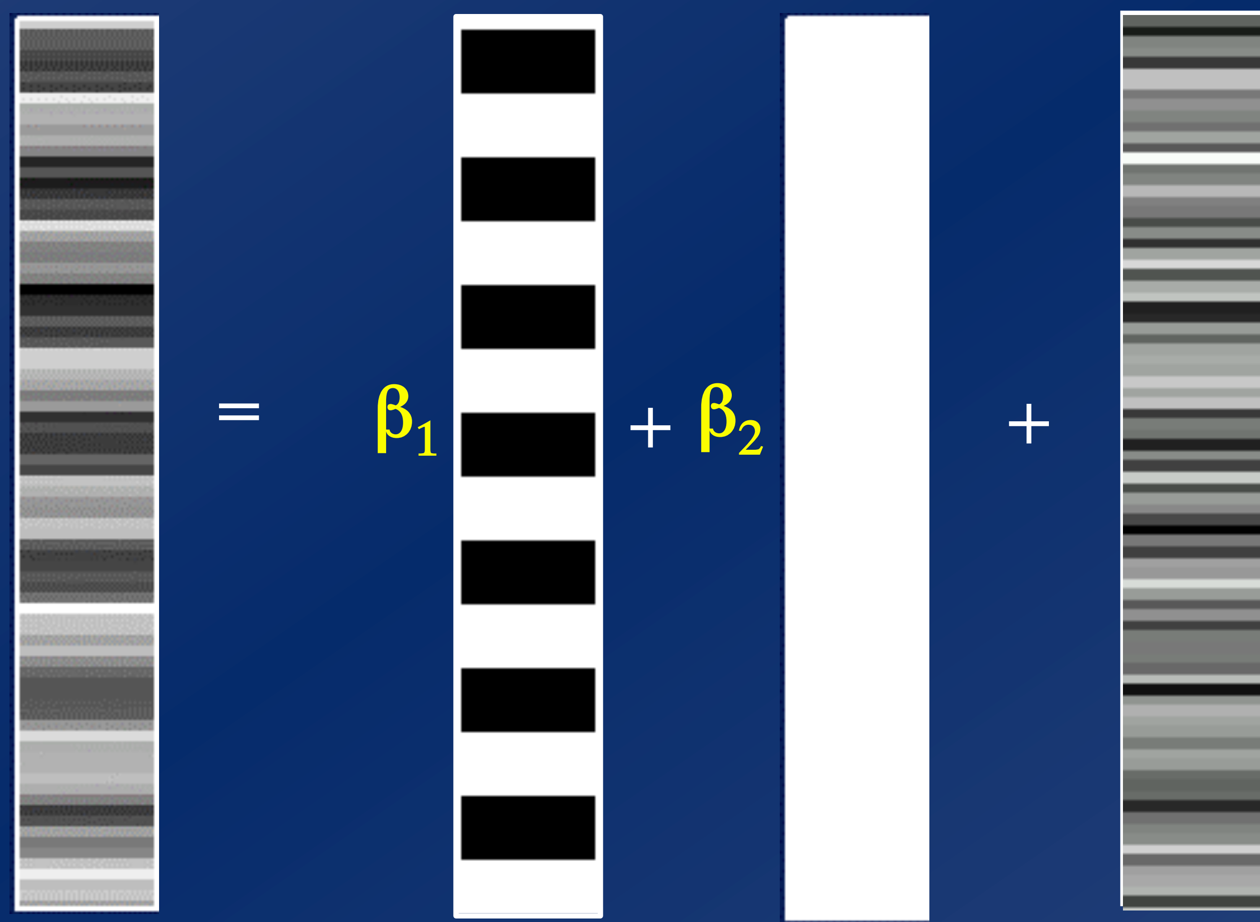
Regression example…



Regression example…

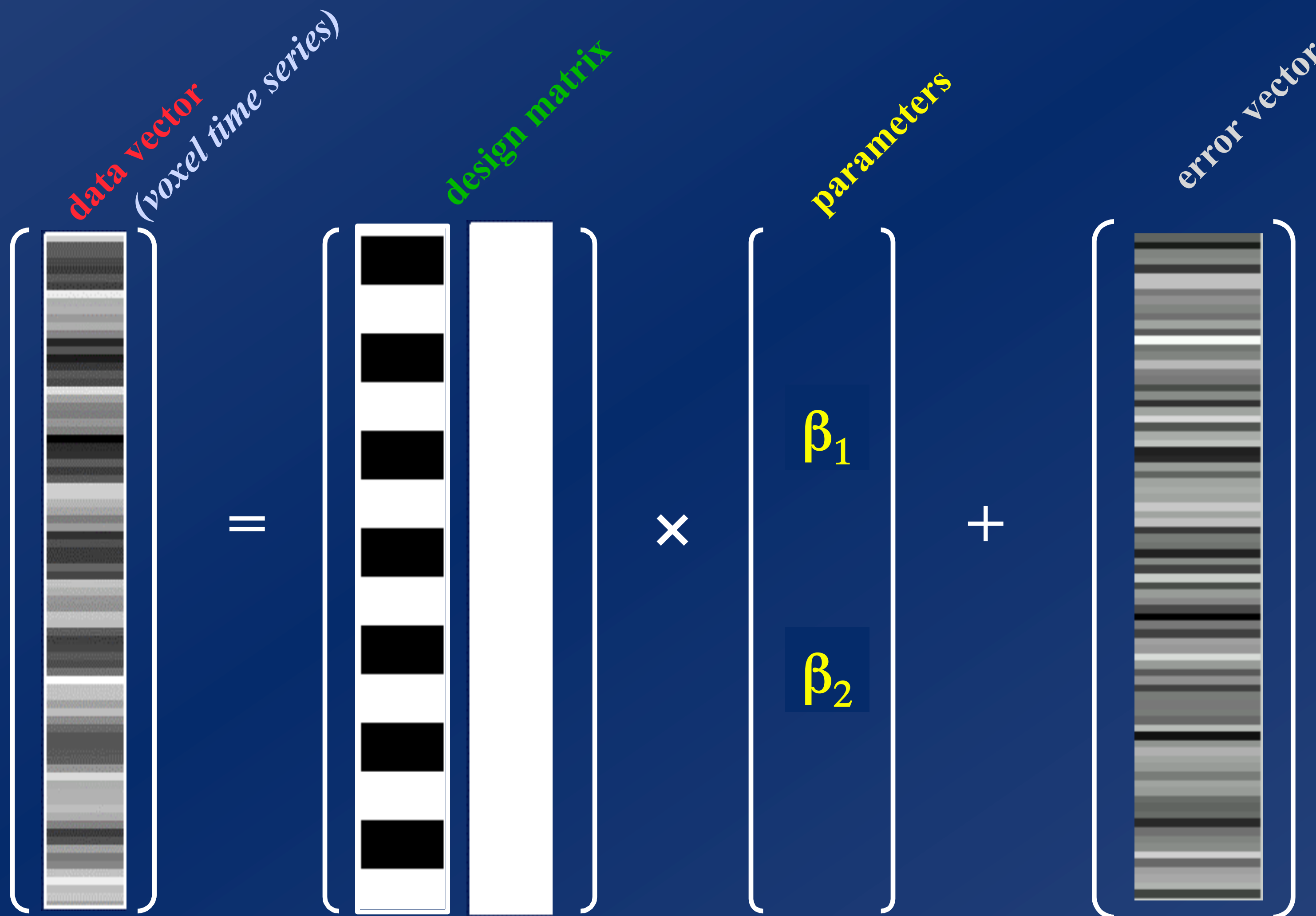


…revisited : matrix form



$$Y = \beta_1 \times f(t) + \beta_2 \times 1 + \varepsilon$$

Box car regression: design matrix…



$$\underline{Y} = X \times \underline{\beta} + \underline{\varepsilon}$$

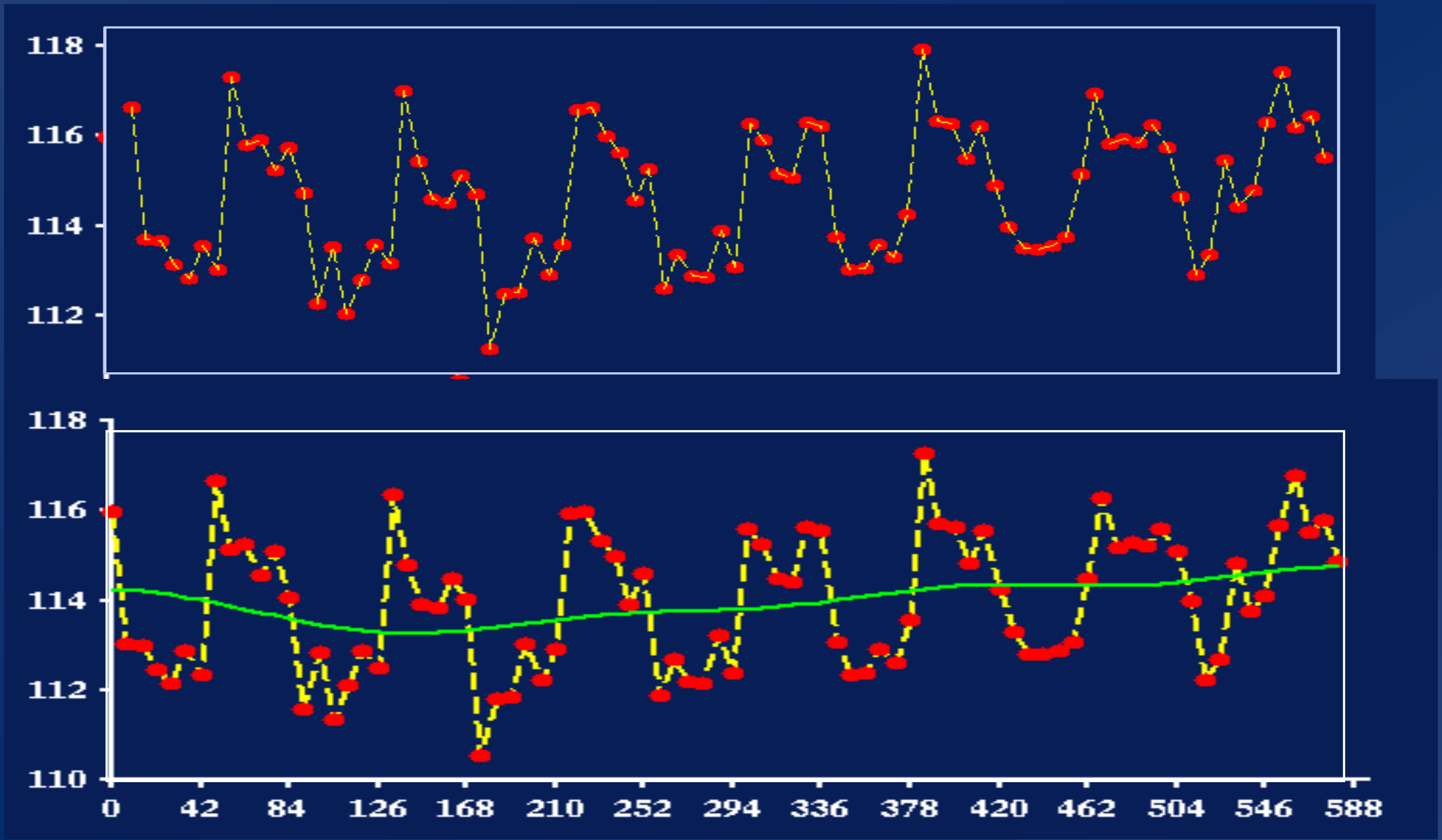
Fact: model parameters depend on regressors scaling

Q: When do I care ?

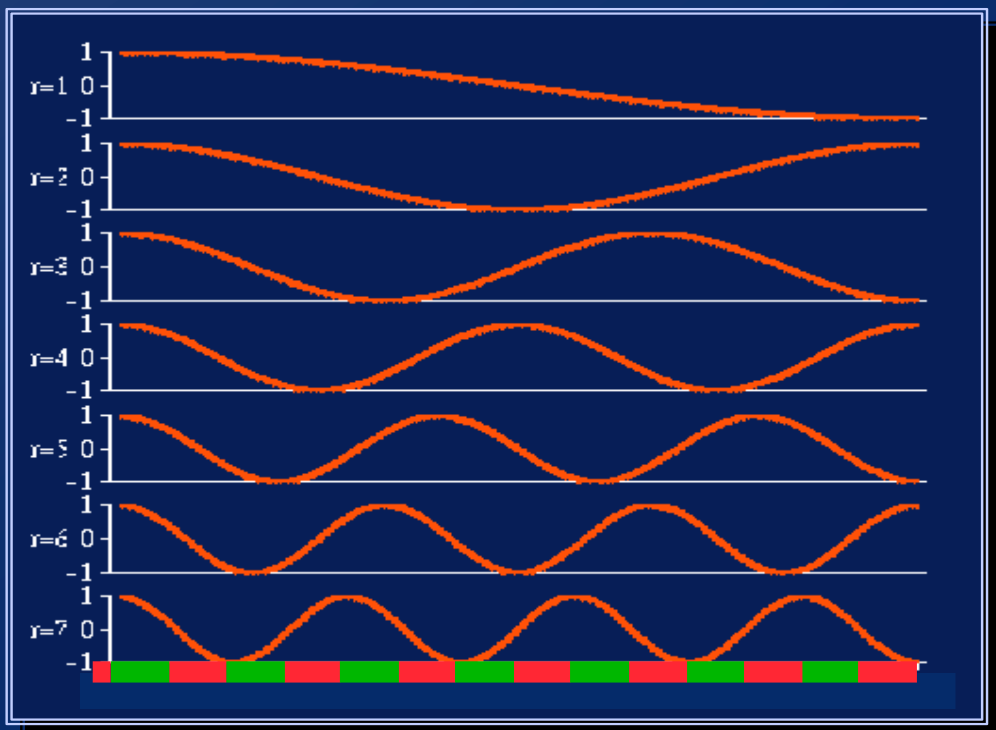
A: ONLY when comparing manually entered regressors (say you would like to compare two scores)

What if two conditions A and B are not of the same duration before convolution HRF?

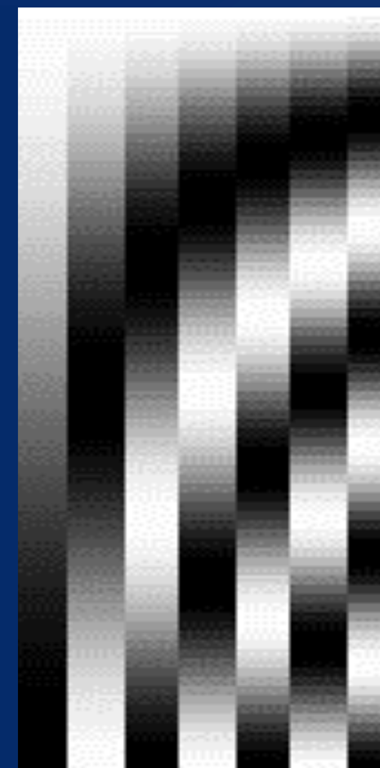
What if we believe that there are drifts?



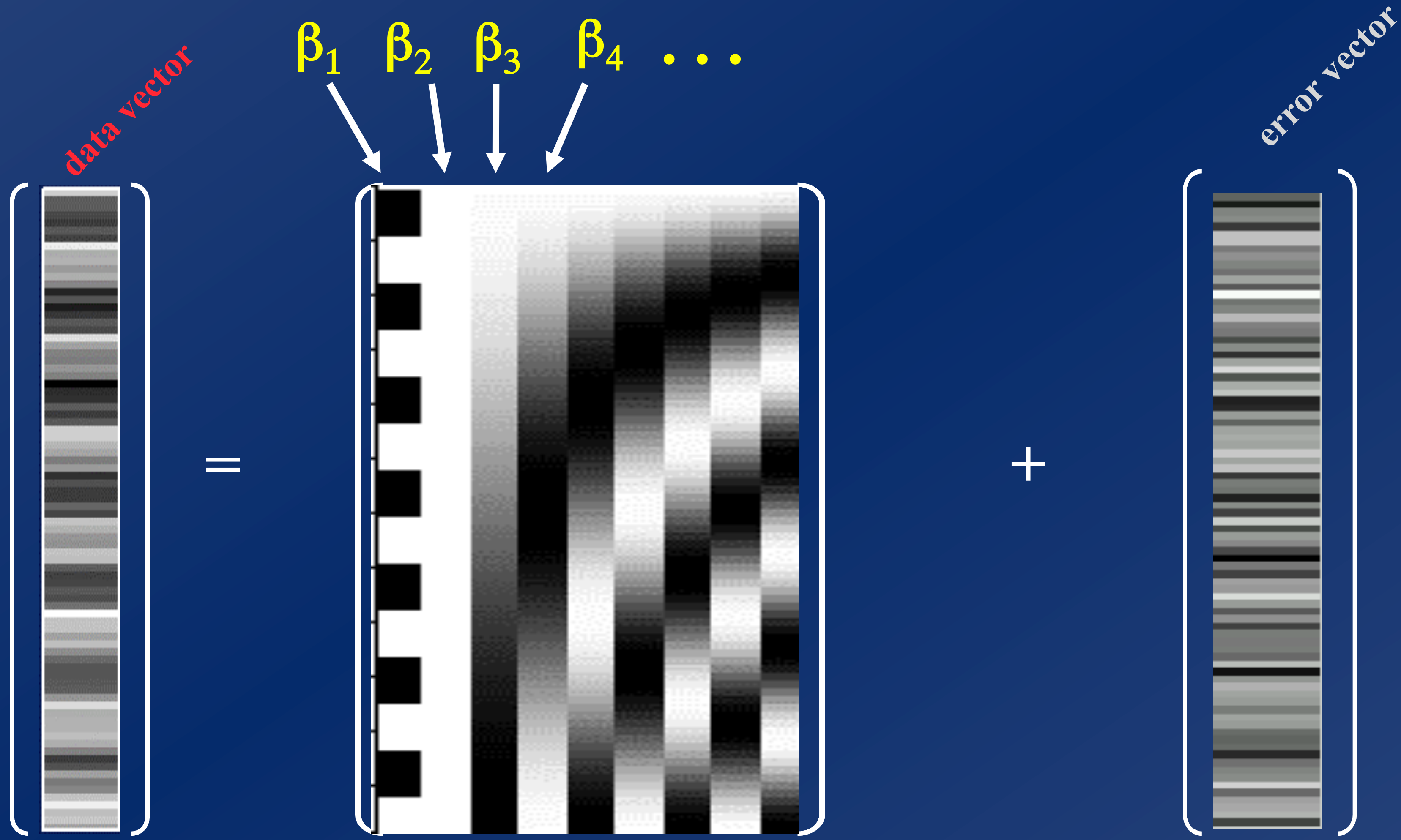
Add more reference functions / covariates ...



Discrete cosine transform basis functions

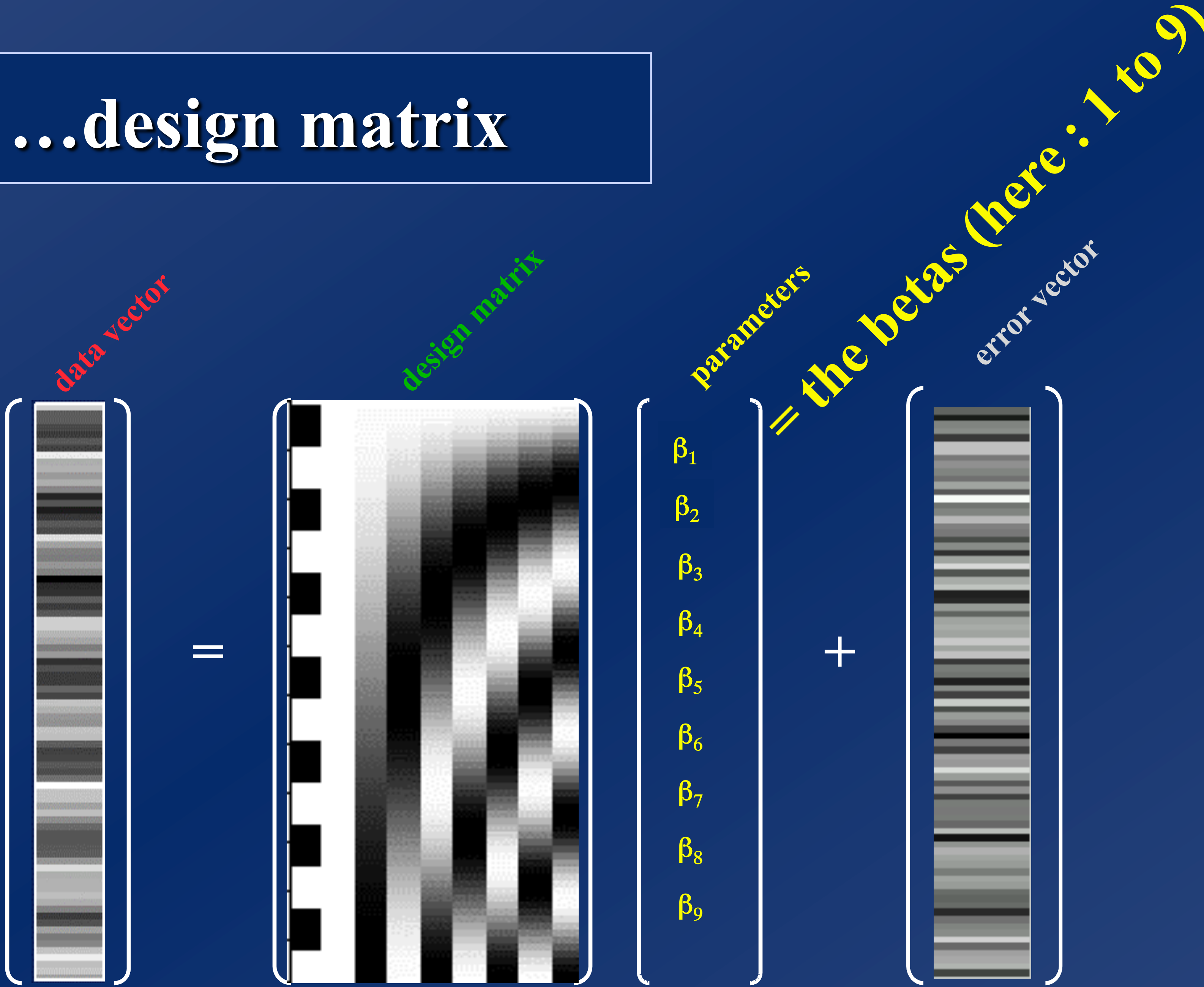


…design matrix



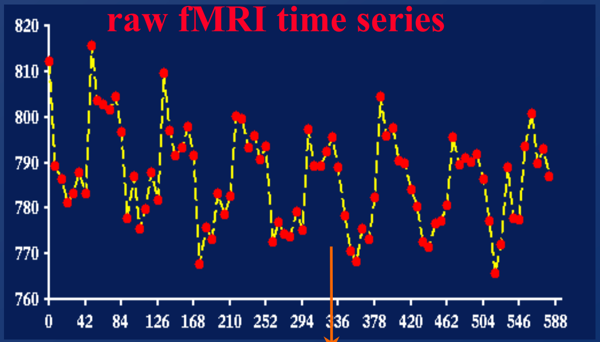
$$Y = X \times \beta + \varepsilon$$

…design matrix

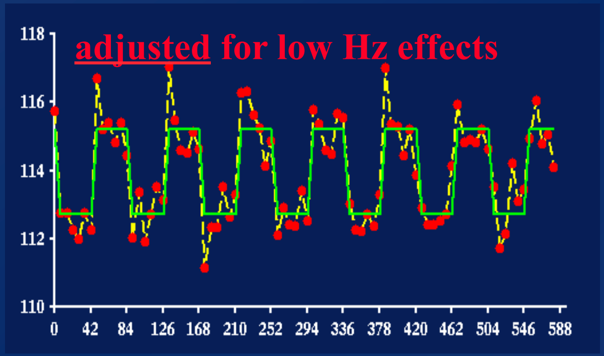


$$Y = X \times \beta + \varepsilon$$

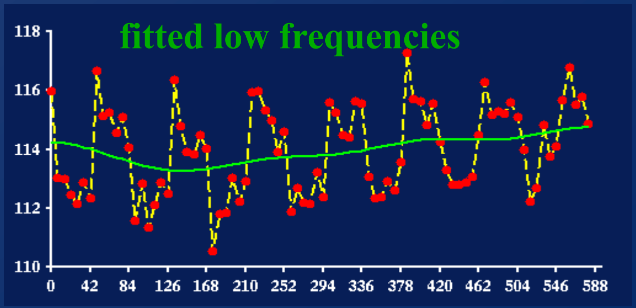
Fitting the model = finding some estimate of the betas



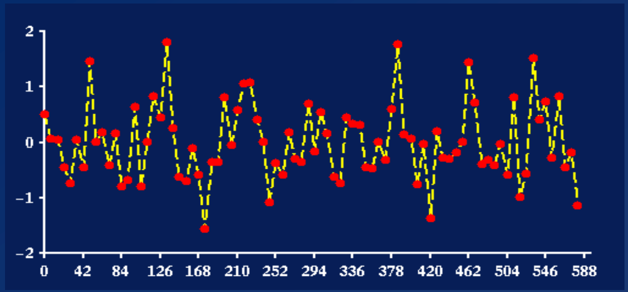
Raw data



fitted signal



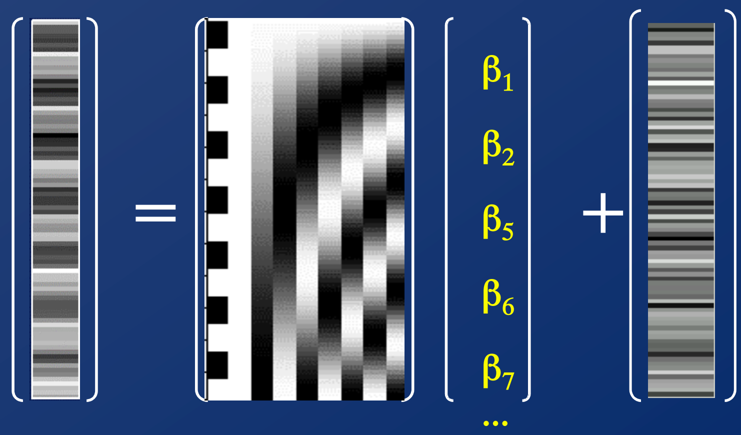
fitted drift



residuals

How do we find the betas estimates? By minimizing the residual variance

Fitting the model = finding some estimate of the betas



$$Y = X\beta + \varepsilon$$

finding the betas = minimising the sum of square of the residuals

$$\| Y - X\hat{\beta} \|^2 = \Sigma_i [y_i - X\hat{\beta}_i]^2$$

when β are estimated: let's call them b (or $\hat{\beta}$)

when ε is estimated : let's call it e

estimated SD of ε : let's call it s

Take home ...

- *We put in our model regressors (or covariates) that represent how we think the signal is varying (of interest and of no interest alike)*
  - ***WHICH ONE TO INCLUDE ?***
  - ***What if we have too many? Too few?***

- *Coefficients (= parameters) are estimated by minimizing the fluctuations, - variability – variance – of estimated noise – the residuals.*

- *Because the parameters depend on the scaling of the regressors included in the model, one should be careful in comparing manually entered regressors, or conditions of different durations*

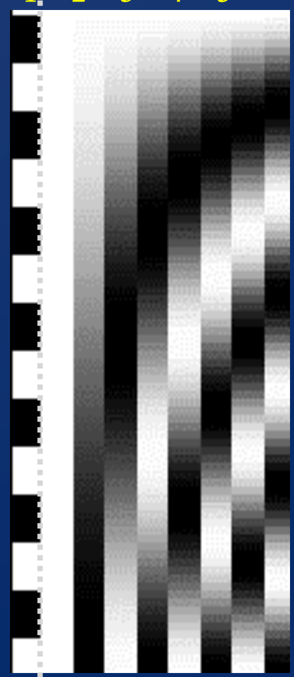
Plan

- *Make sure we all know about the estimation (fitting) part ....*
- *Make sure we understand t and F tests*
- *But what do we test exactly ?*
- *An example – almost real*

T test - one dimensional contrasts - SPM{*t*}

$c' = 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0$

$b_1\ b_2\ b_3\ b_4\ b_5\ ....$



A contrast = a weighted sum of parameters: c´ × b

$b_1 > 0$?

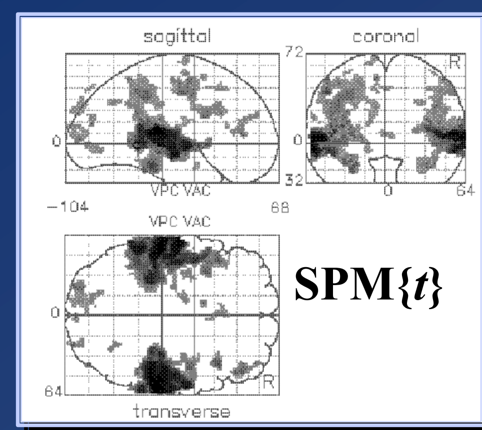
Compute $1 \text{x} b_1 + 0 \text{x} b_2 + 0 \text{x} b_3 + 0 \text{x} b_4 + 0 \text{x} b_5 + \ldots = c'b$

$c' = [1\ 0\ 0\ 0\ 0\ \ldots.]$

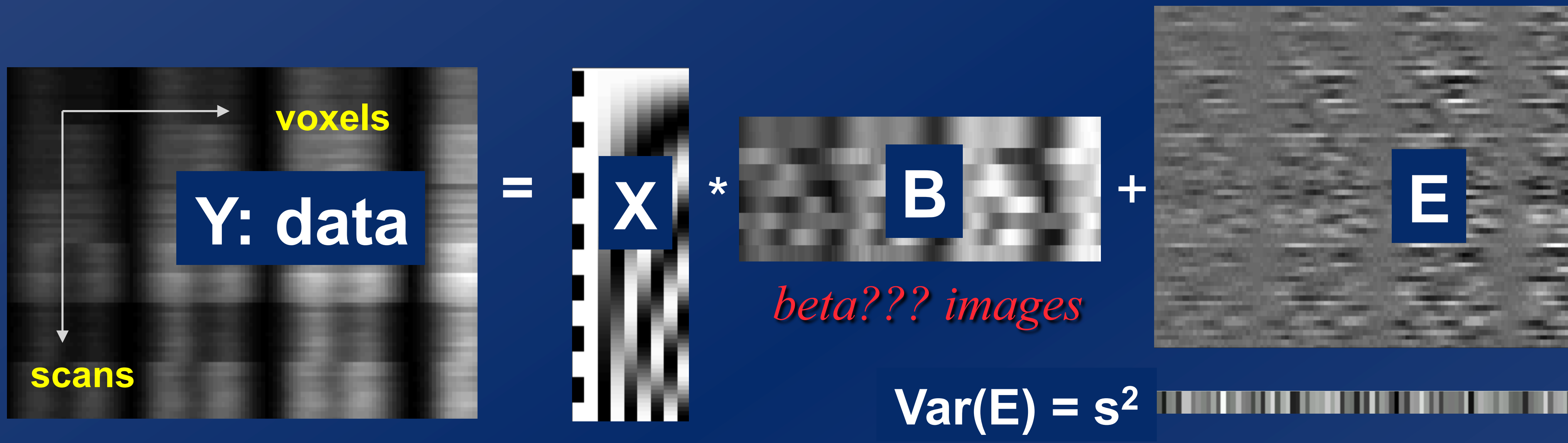
divide by estimated standard deviation of b_1

$$T = \frac{\textit{contrast} \text{ of estimated parameters}}{\sqrt{\text{variance estimate}}}$$

$$T = \frac{c'b}{\sqrt{s^2 c'(X'X)^- c}}$$

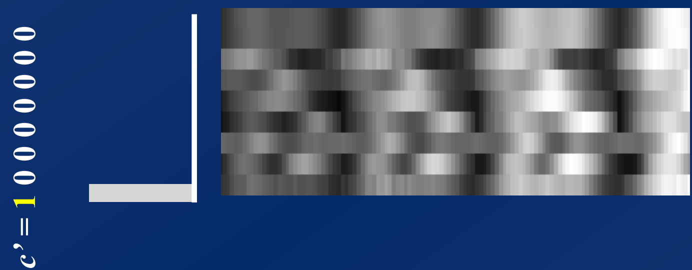


From one time series to an image



beta??? images

spm_ResMS



$$T = \frac{c'b}{\sqrt{s^2 c'(X'X)^- c}} =$$

spm_con??? images

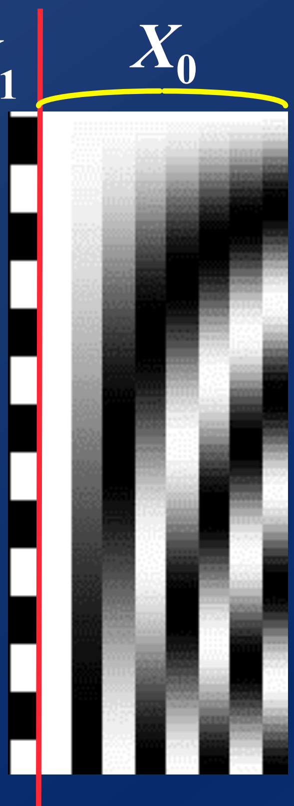
spm_t??? images

F-test : a reduced model

H_0: True **model is** X_0

H_0: $\beta_1 = 0$

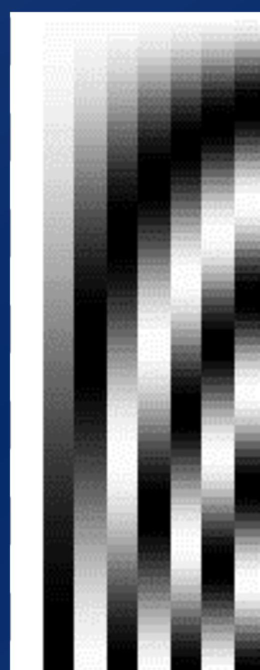
X_1 X_0



$\rightarrow S^2$

This (full) model ?

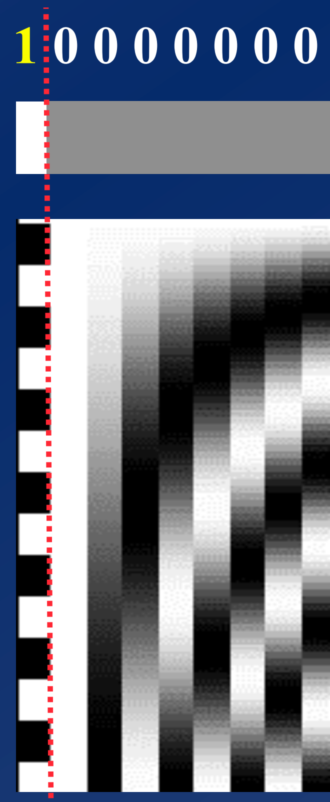
X_0



$\rightarrow S_0^2$

Or this one?

$c' = 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0$



$F \sim (S_0^2 - S^2) / S^2$

T values become F values. F = T²

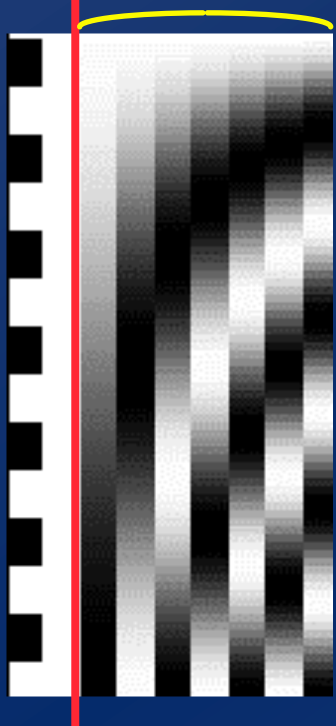
Both "activation" and "deactivations" are tested. Voxel wise p-values are halved.

F-test : a reduced model or ...

Tests multiple linear hypotheses : Does X1 model anything ?

H$_0$: True (reduced) model is X_0

X_0 X_1



$\longrightarrow$ S^2

This (full) model ?

X_0

$\longrightarrow$ S_0^2

Or this one?

$$F = \frac{\text{additional variance accounted for by tested effects}}{\text{error variance estimate}}$$

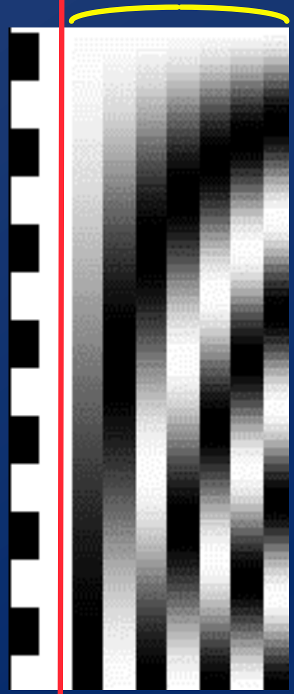
$$F \sim (S_0^2 - S^2) / S^2$$

F-test : a reduced model or ... multi-dimensional contrasts ?

tests multiple linear hypotheses. Ex : does drift functions model anything?

H_0: True model is X_0

X_0 X_1

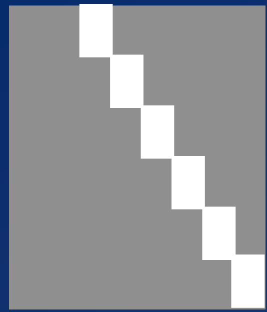


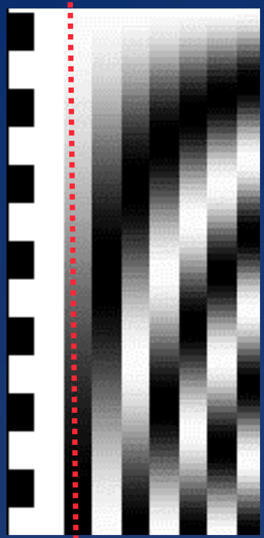
X_0

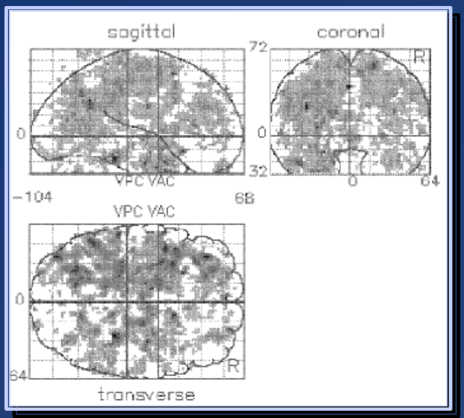
This (full) model ? Or this one?

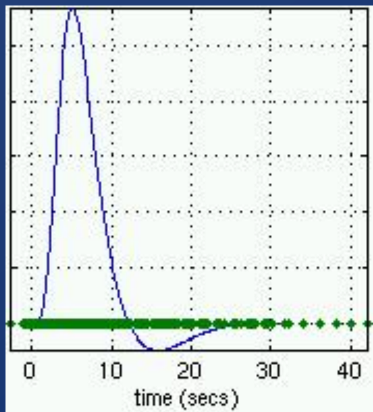
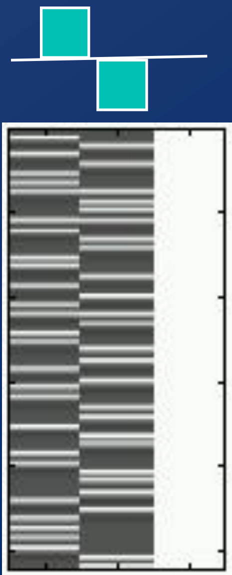
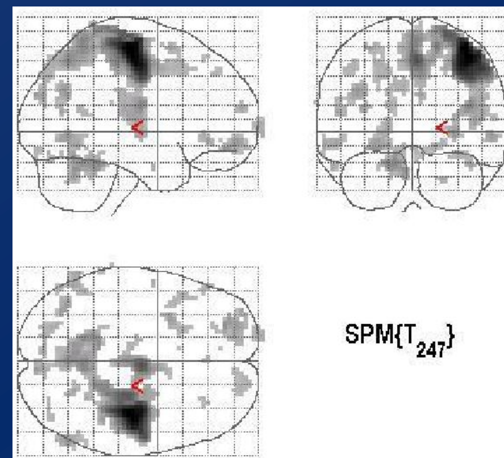
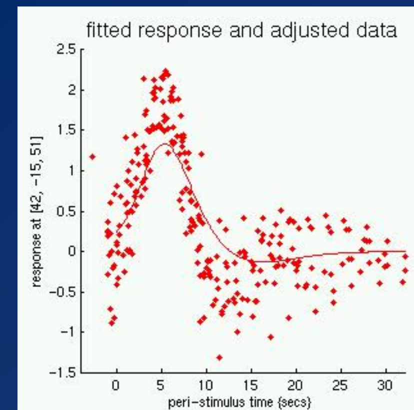
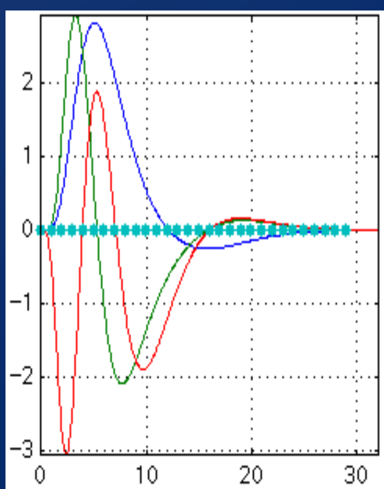
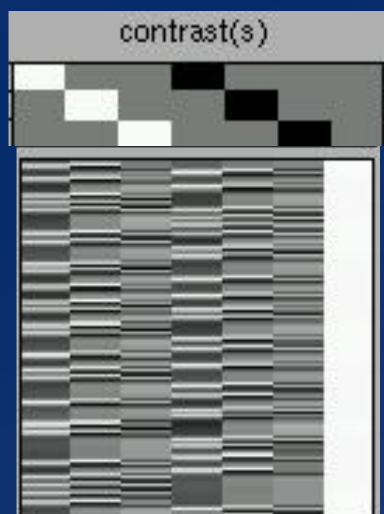
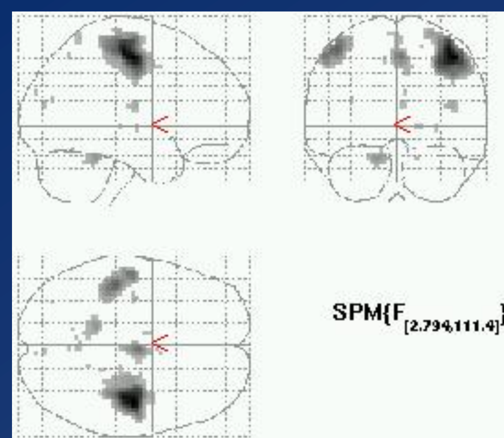
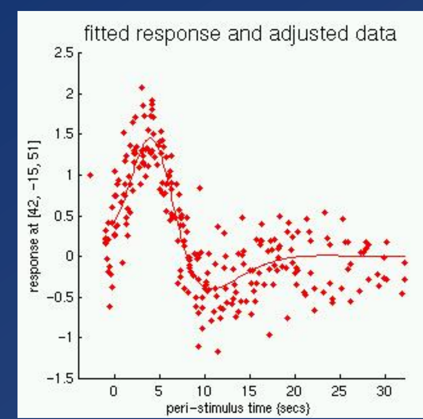
H_0: $\beta_{3-9} = (0\ 0\ 0\ 0\ ...)$

$$c' = \begin{matrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}$$







Convolution model	Design and contrast	SPM(t) or SPM(F)	Fitted and adjusted data
			
			

T and F test: take home ...

- *T tests are simple combinations of the betas; they are either positive or negative (b1 – b2 is different from b2 – b1)*

- *F tests can be viewed as testing for the additional variance explained by a larger model wrt a simpler model, or*

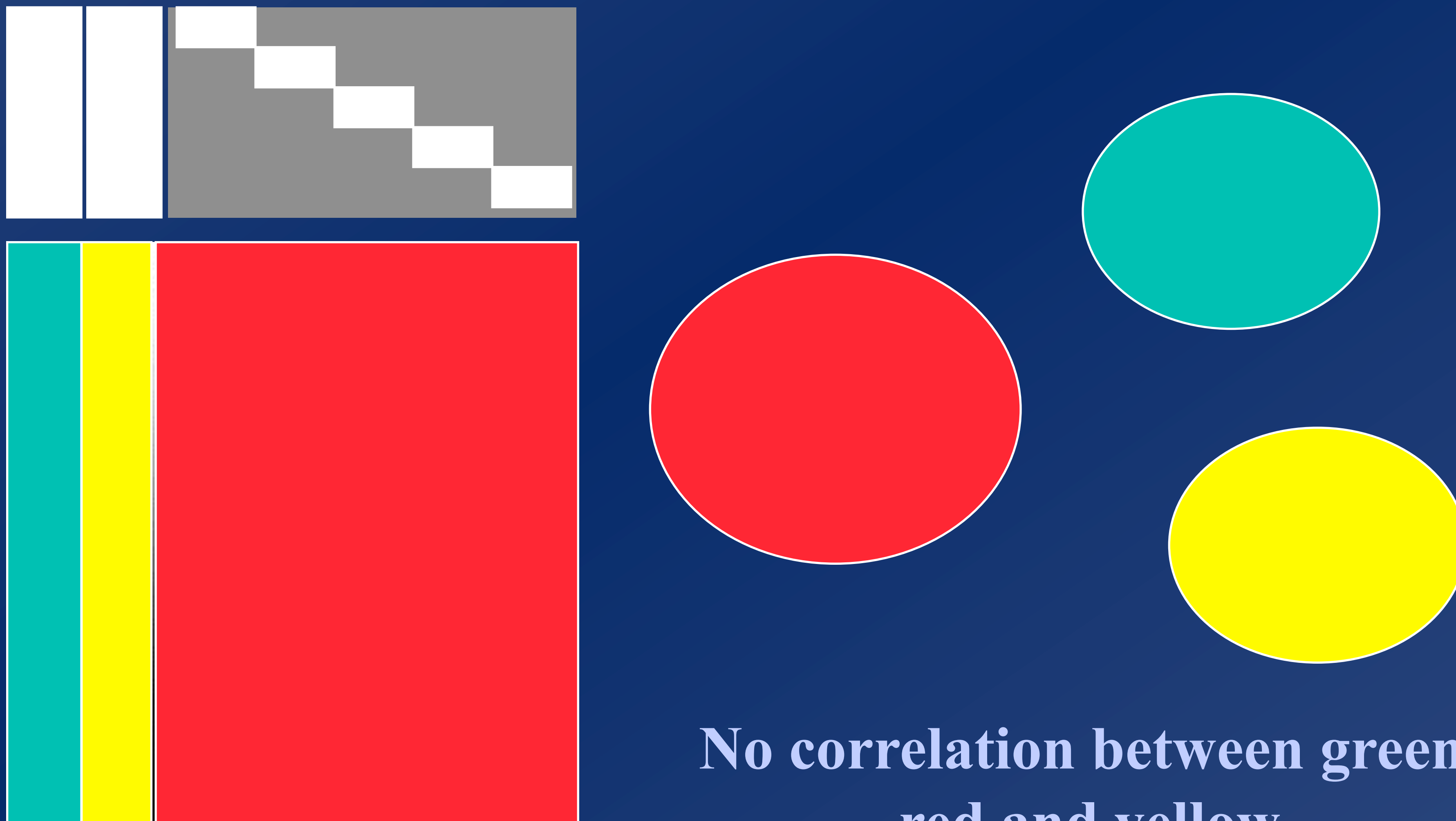
- *F tests the sum of the squares of one or several combinations of the betas*

- *in testing "single contrast" with an F test, for ex. b1 – b2, the result will be the same as testing b2 – b1. It will be exactly the square of the t-test, testing for both positive and negative effects.*

Plan

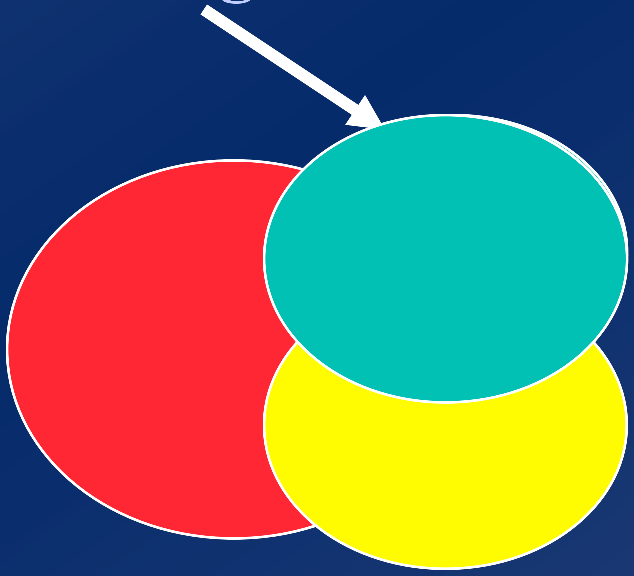
- *Make sure we all know about the estimation (fitting) part ....*
- *Make sure we understand t and F tests*
- *But what do we test exactly ? Correlation between regressors*
- *An example – almost real*

« Additional variance » : Again

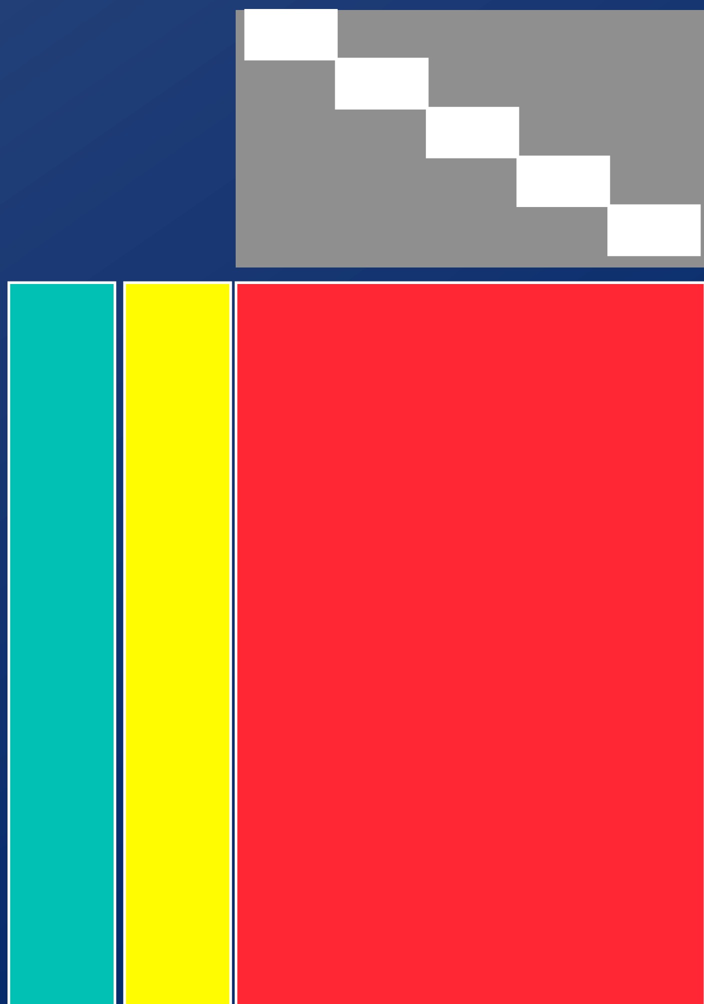


No correlation between green red and yellow

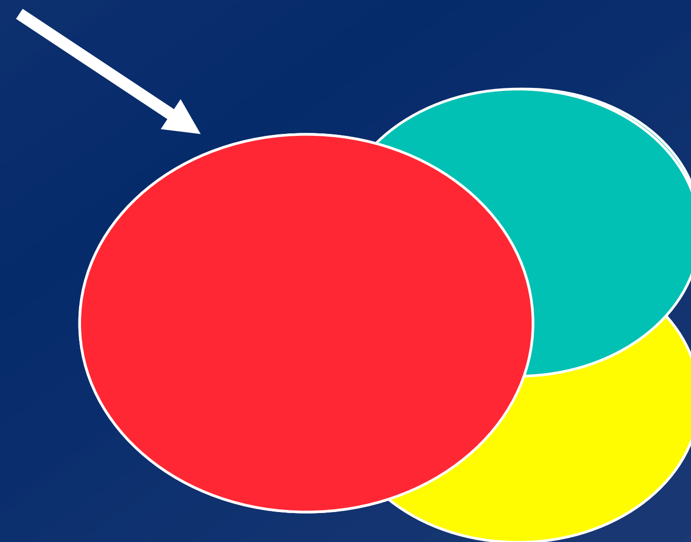
Testing for the green



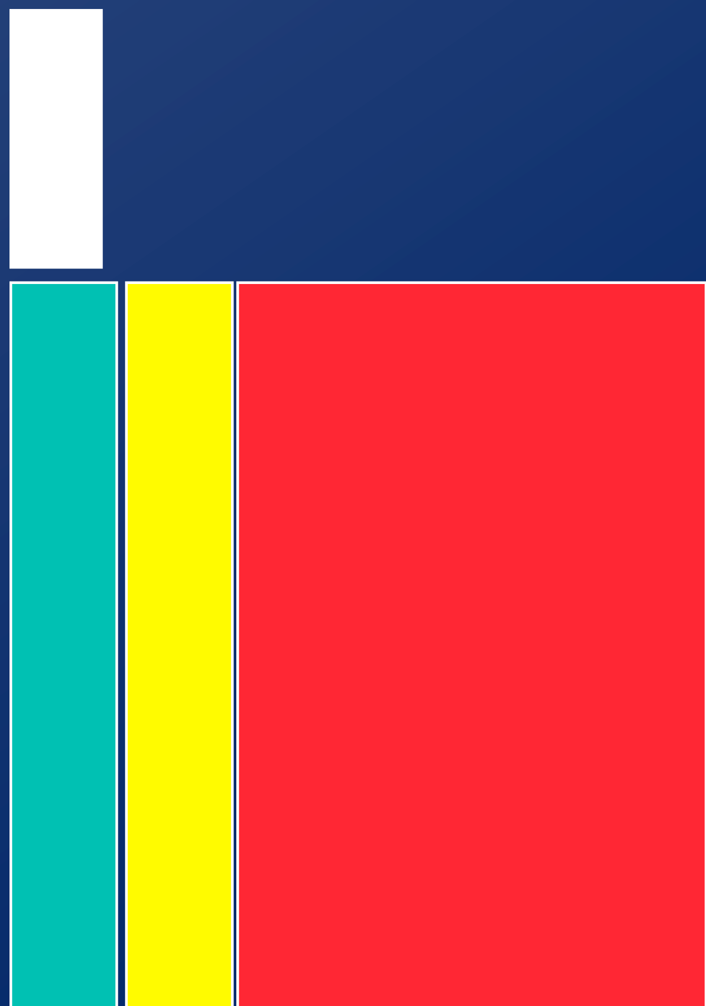
correlated regressors, for example
green: subject age
yellow: subject score



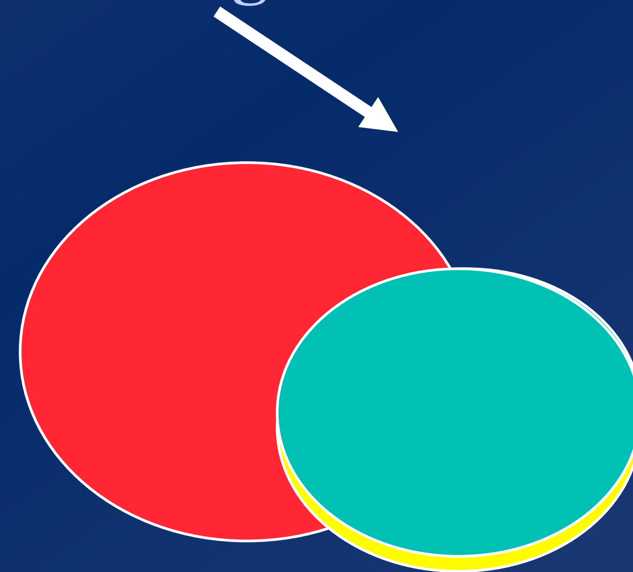
Testing for the red



correlated contrasts

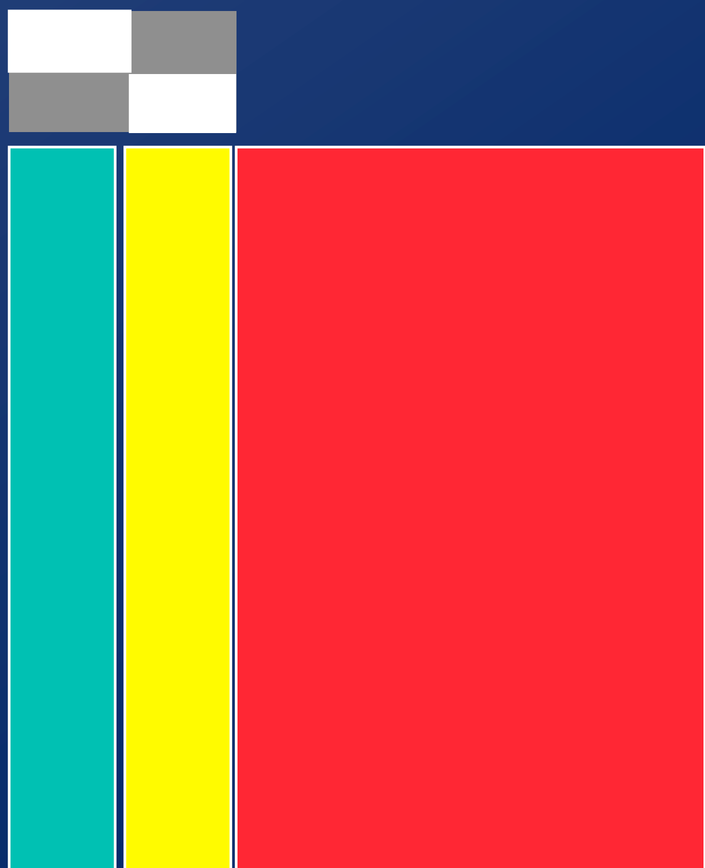


Testing for the green

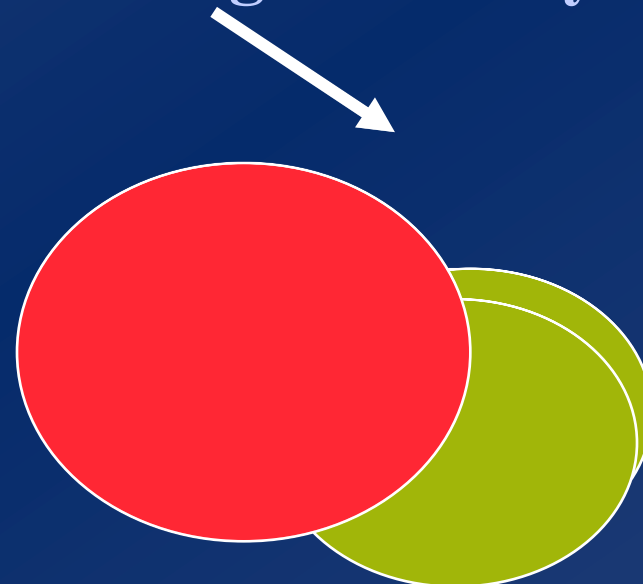


Very correlated regressors ?

Dangerous !

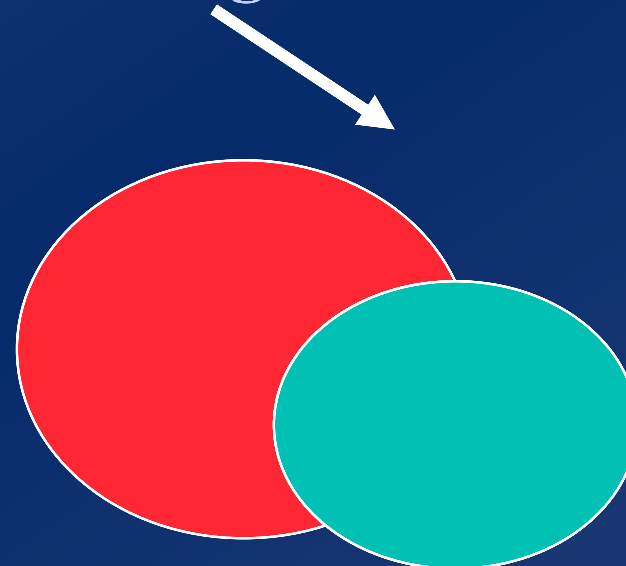


Testing for the green and yellow



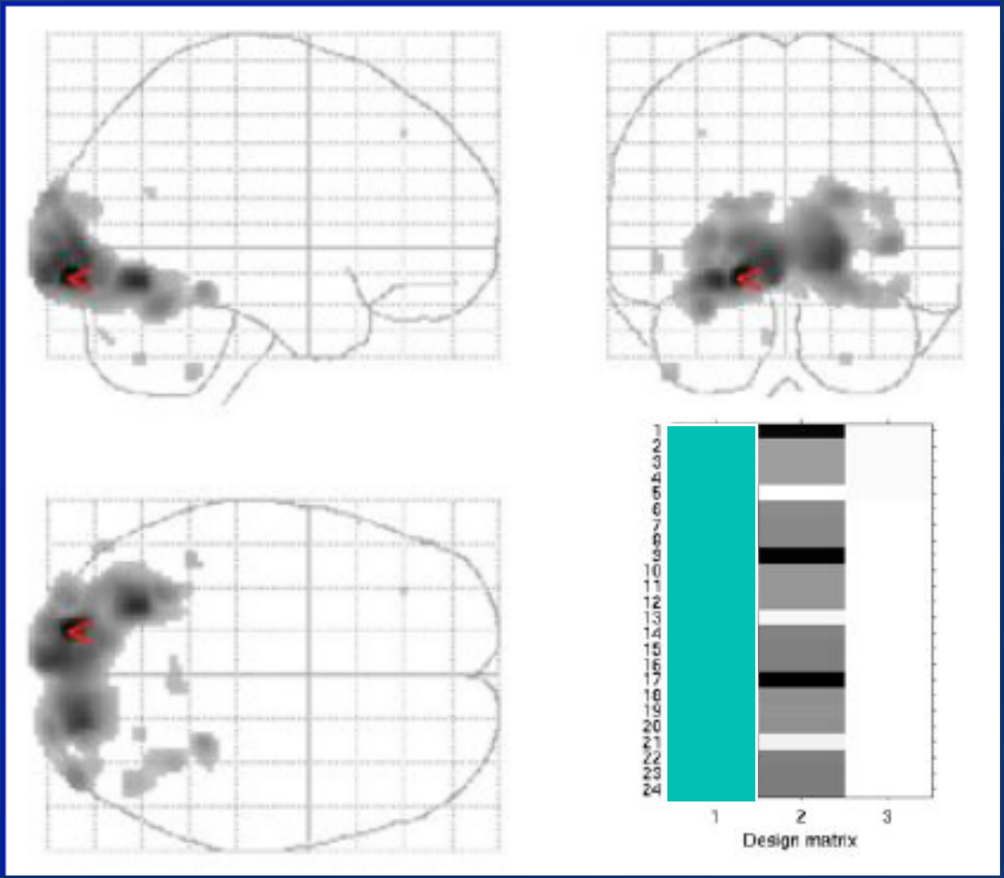
If significant ? Could be G or Y !

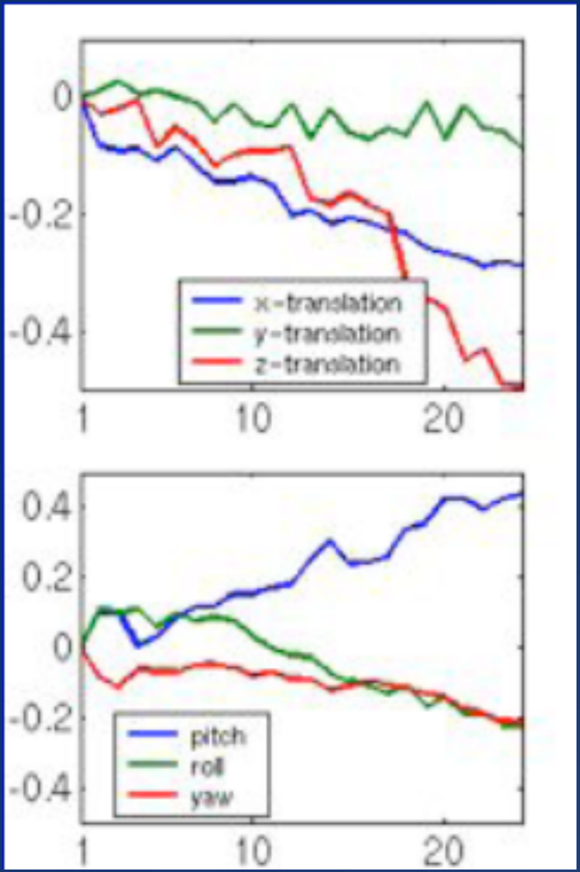
Testing for the green



Completely correlated regressors ?
Impossible to test ! (not estimable)

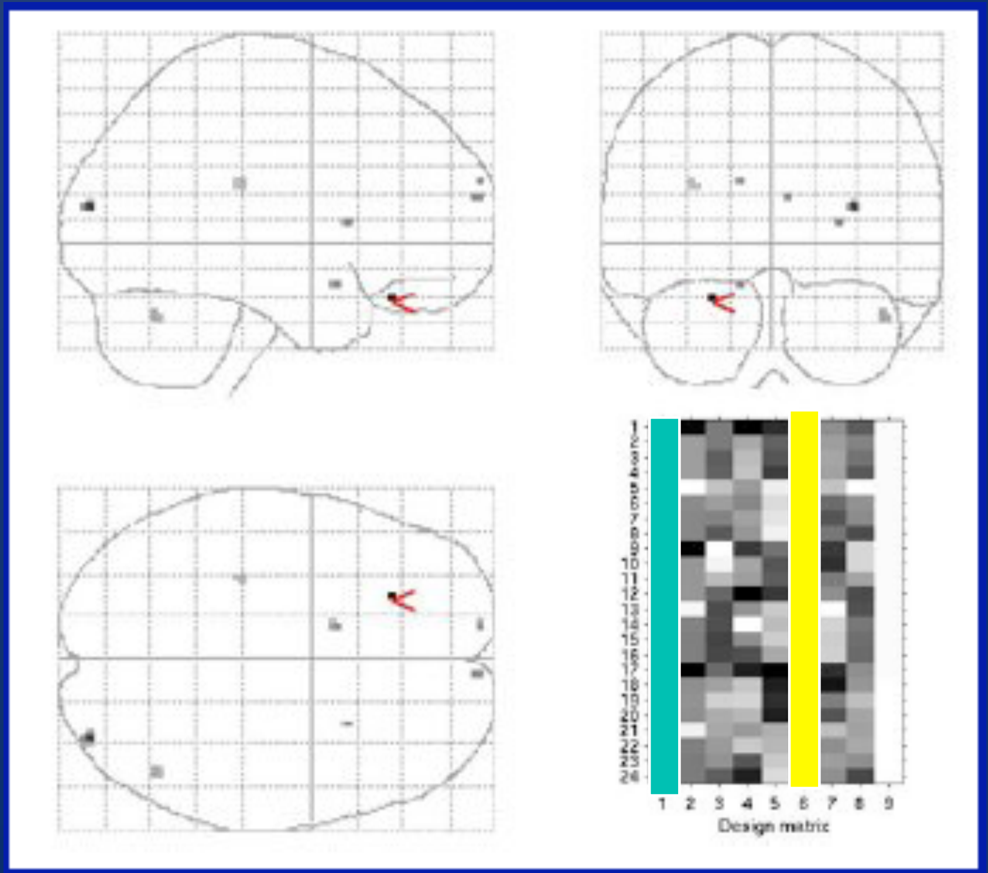
An example: real

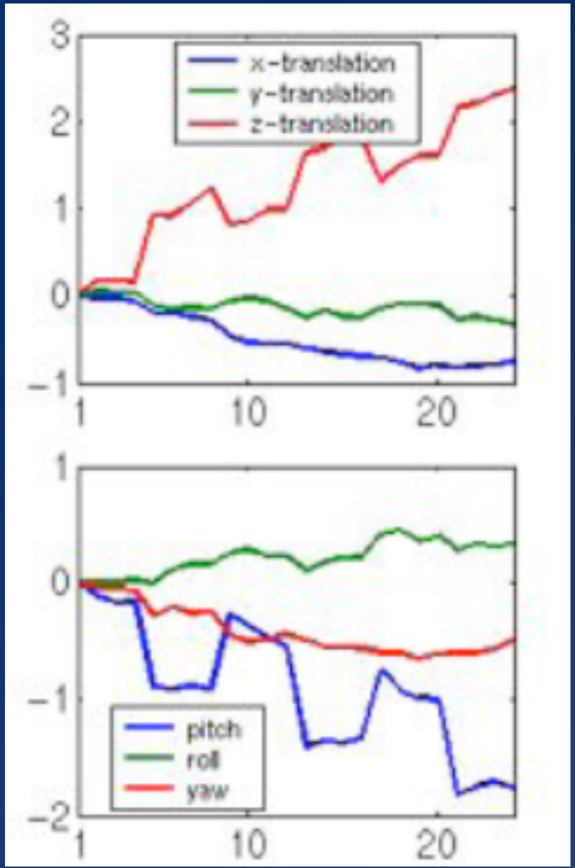




Testing for first regressor: T max = 9.8

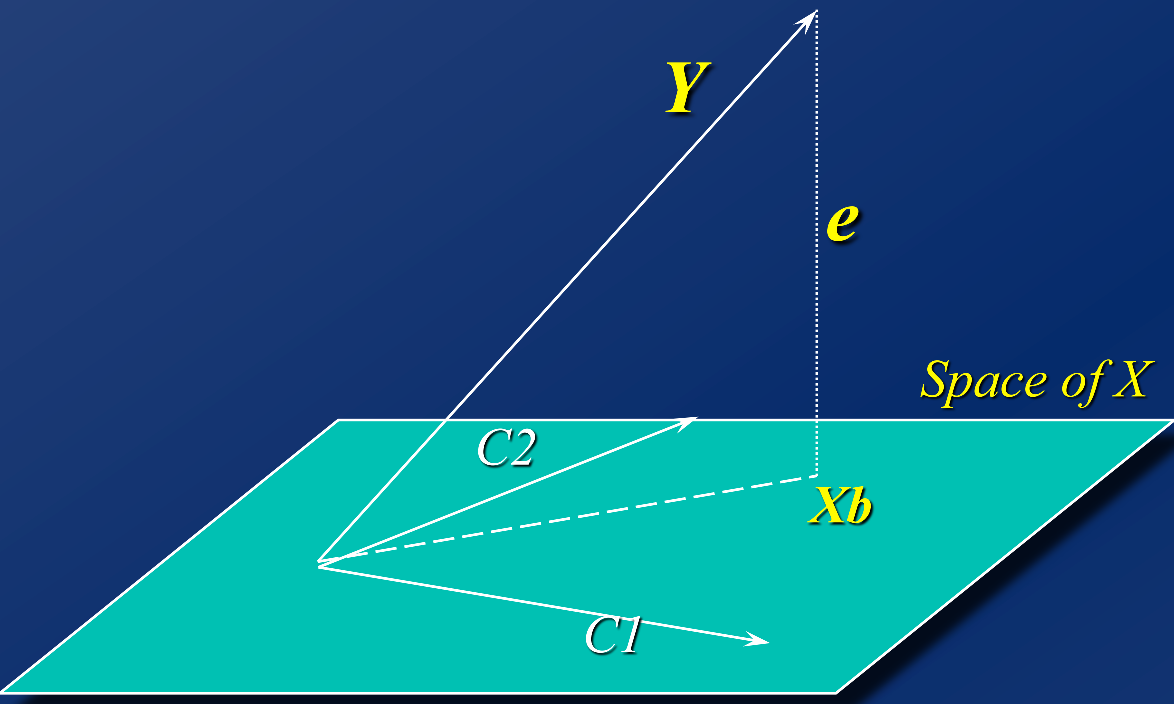
Including the movement parameters in the model

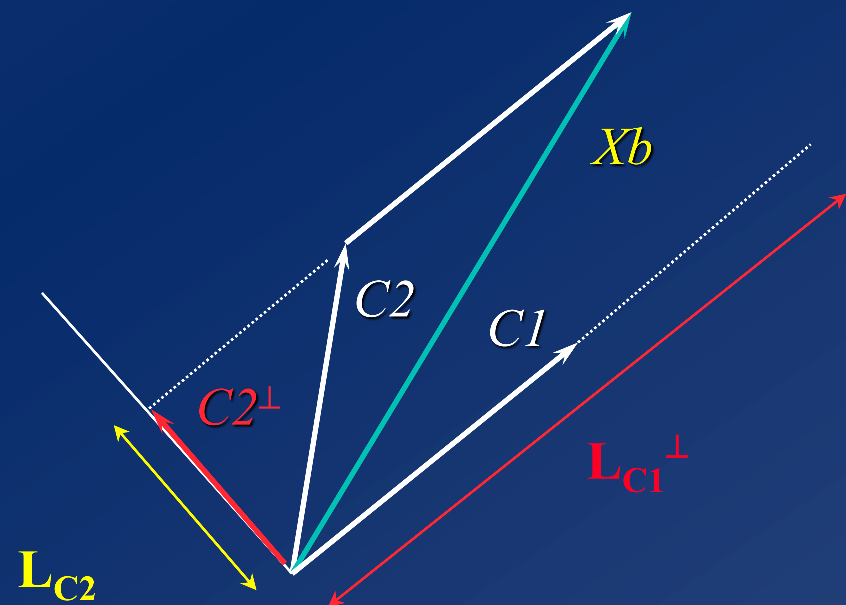




Testing for first regressor: activation is gone !
Right or Wrong?

Implicit or explicit (⊥) decorrelation (or orthogonalisation)





This generalises when testing several regressors (F tests)

L_{C2} : test of C2 in the implicit ⊥ model

$L_{C1}^{\perp}$: test of C1 in the explicit ⊥ model

cf Andrade et al., NeuroImage, 1999

Correlation between regressors: take home ...

- *Do we care about correlation in the design ? Yes, always*

- *Start with the experimental design : conditions should be as uncorrelated as possible*

- *use F tests to test for the overall variance explained by several (correlated) regressors*

Plan

- *Make sure we all know about the estimation (fitting) part ....*
- *Make sure we understand t and F tests*
- *But what do we test exactly ? Correlation between regressors*
- *An example – almost real*

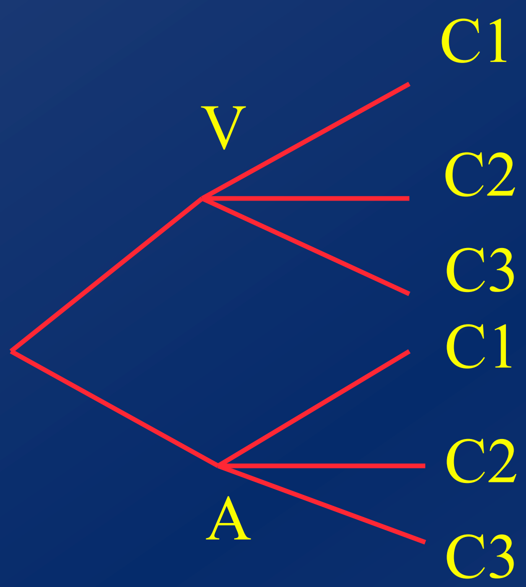
A real example (almost !)

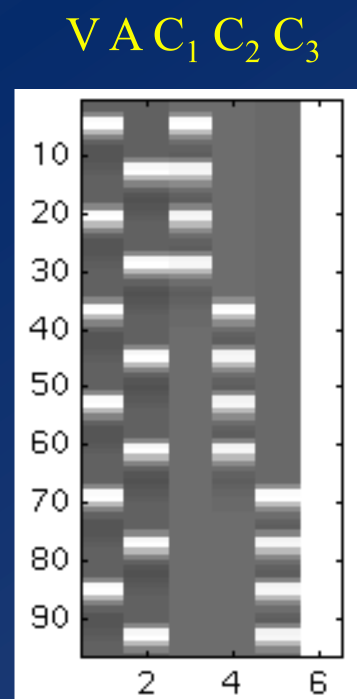
Experimental Design → Design Matrix

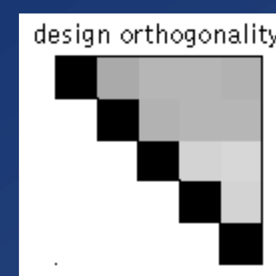
Factorial design with 2 factors : modality and category

2 levels for modality (eg Visual/Auditory)

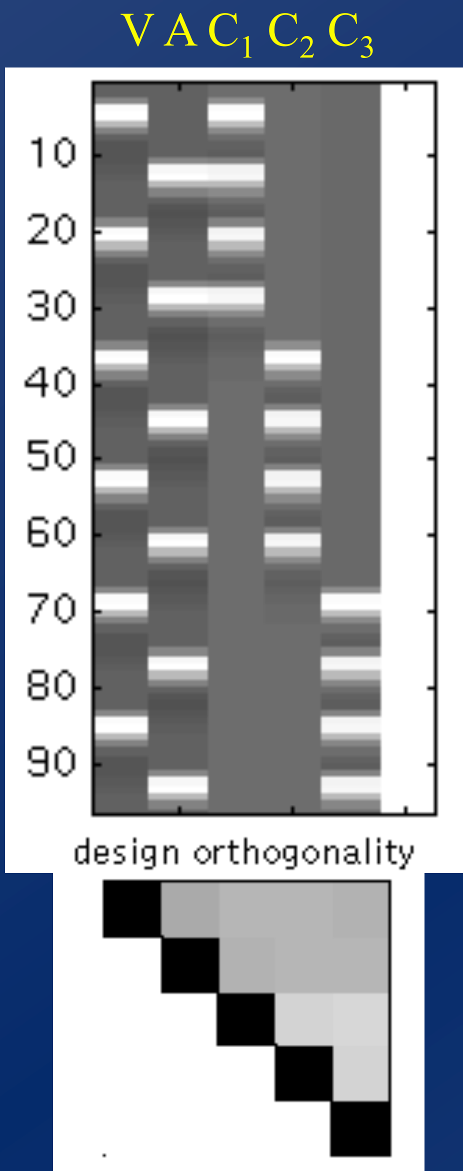
3 levels for category (eg 3 categories of words)







Asking ourselves some questions ...



Test C1 > C2 : c = [0 0 1 -1 0 0]

Test V > A : c = [1 -1 0 0 0 0]

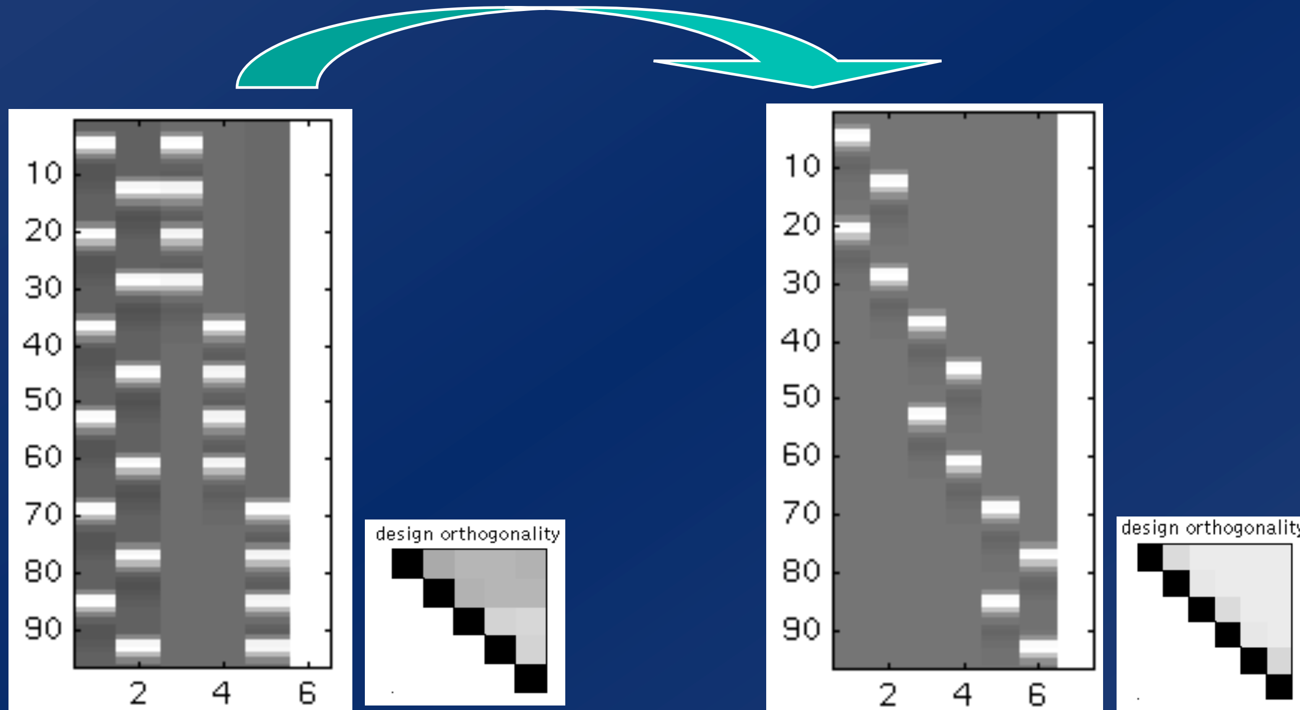
Test C1,C2,C3 ? (F)

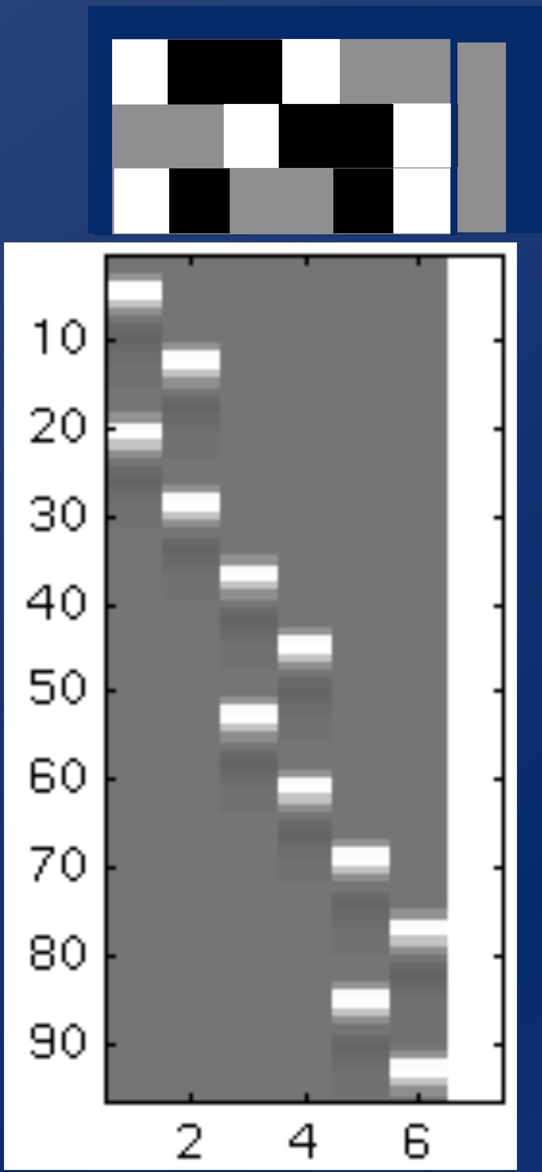
c =
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[0 0 0 0 1 0]

Test the interaction MxC ?

- Design Matrix not orthogonal
- Many contrasts are non estimable
- Interactions MxC are not modelled

Modelling the interactions





Test C1 > C2 : c = [1 1 -1 -1 0 0 0]

Test V > A : c = [1 -1 1 -1 1 -1 0]

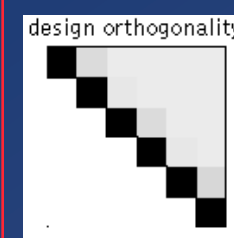
Test the category effect :

c =
[1 1 -1 -1 0 0 0]
[0 0 1 1 -1 -1 0]
[1 1 0 0 -1 -1 0]

Test the interaction MxC :

c =
[1 -1 -1 1 0 0 0]
[0 0 1 -1 -1 1 0]
[1 -1 0 0 -1 1 0]

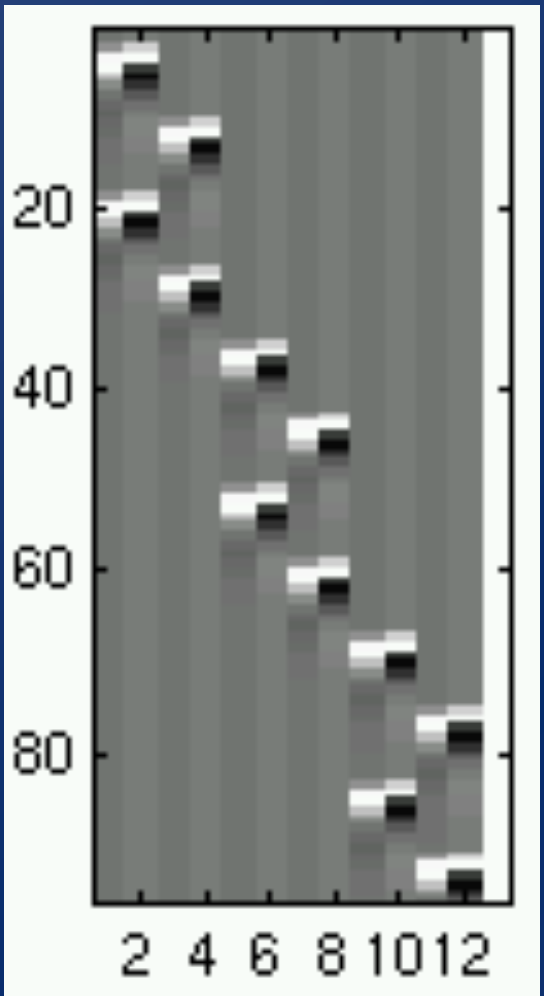
- Design Matrix orthogonal
- All contrasts are estimable
- Interactions MxC modelled
- If no interaction ... ? Model is too "big" !



With a more flexible model

C_1 C_1 C_2 C_2 C_3 C_3

V A V A V A



Test C1 > C2 ?

Test C1 different from C2 ?

from

c = [1 1 -1 -1 0 0 0]

to

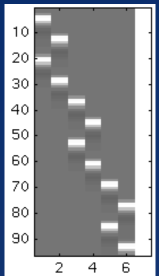
c = [1 0 1 0 -1 0 -1 0 0 0 0 0 0]

[0 1 0 1 0 -1 0 -1 0 0 0 0 0]

becomes an F test!

What if we use only:

c = [1 0 1 0 -1 0 -1 0 0 0 0 0 0]



OK only if the regressors coding for the delay are all equal

Toy example: take home ...

- *use F tests when*
  - *Test for >0 and <0 effects*
  - *Test for more than 2 levels in factorial designs*
  - *Conditions are modelled with more than one regressor*

- *Check post hoc*

Thank you for your attention!

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Design Matrix	Parameters	Contrasts
(1) $X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{cases} \hat{\beta}_1 = \bar{y_1} \\ \hat{\beta}_2 = \bar{y_2} \end{cases}$	$(1,0).\hat{\beta} = \bar{y_1}$ $(0,1).\hat{\beta} = \bar{y_2}$ $(1,-1).\hat{\beta} = \bar{y_1} - \bar{y_2}$ $(.5,.5).\hat{\beta} = \text{mean}(\bar{y_1}, \bar{y_2})$

$$P_X Y = X \beta$$

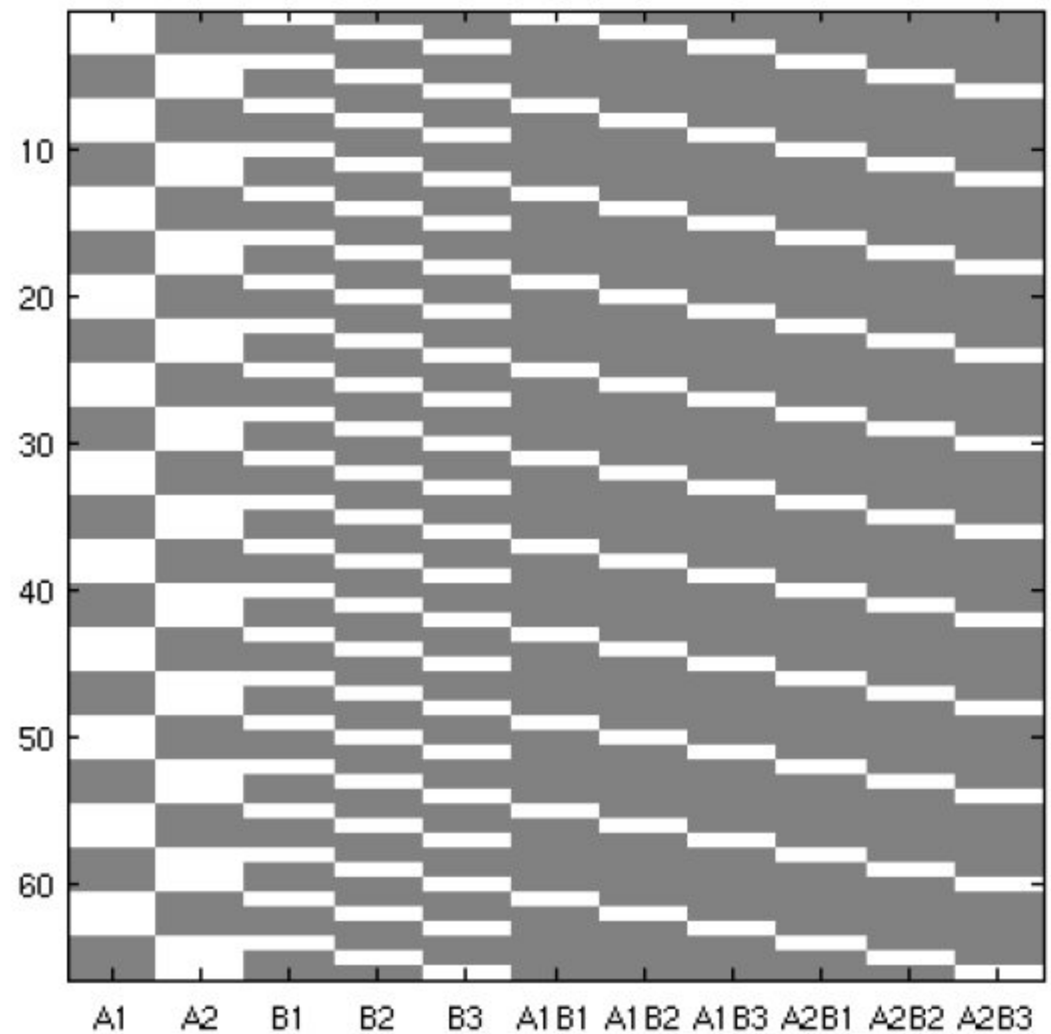
Projector onto X

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$P_X Y = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix} .Y = X\beta = \begin{bmatrix} \bar{y_1} \\ \bar{y_1} \\ \bar{y_2} \\ \bar{y_2} \\ \bar{y_2} \end{bmatrix}$$

$$(2) \quad X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \begin{cases} \hat{\beta}_1 + \hat{\beta}_2 = \bar{y_1} \\ \hat{\beta}_2 = \bar{y_2} \end{cases} \qquad \begin{array}{l} (1,1).\hat{\beta} = \bar{y_1} \\ (0,1).\hat{\beta} = \bar{y_2} \\ (1,0).\hat{\beta} = \bar{y_1} - \bar{y_2} \\ (.5,1).\hat{\beta} = \text{mean}(\bar{y_1}, \bar{y_2}) \end{array}$$

$$(3) \quad X = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{cases} \hat{\beta}_1 + \hat{\beta}_3 = \bar{y_1} \\ \hat{\beta}_2 + \hat{\beta}_3 = \bar{y_2} \end{cases} \qquad \begin{array}{l} (1,0,1).\hat{\beta} = \bar{y_1} \\ (0,1,1).\hat{\beta} = \bar{y_2} \\ (1,-1,0).\hat{\beta} = \bar{y_1} - \bar{y_2} \\ (.5,.5,1).\hat{\beta} = \text{mean}(\bar{y_1}, \bar{y_2}) \end{array}$$


10
20
30
40
50
60
A1
A2
B1
B2
B3
A1B1
A1B2
A1B3
A2B1
A2B2
A2B3

Main Effects and Interaction:

1. Main effect: 2 (**A**)
2. Main effect: 3 (**B**)
3. Interaction: 2 3 (**A** × **B**)

Contrast Weights

1. Main effect of **A**: 1 -1 0 0 0 ones(1,3)/3 -ones(1,3)/3
2. Main effect of **B**: 0 0 -1 0 1 [-1 0 1]*[1/2] [-1 0 1]*[1/2]
3. Interaction **A** × **B**: 0 0 0 0 0 -1 0 1 1 0 -1
4. Test for a single regressor in main effect of **A** (e.g. A1)
   1 0 ones(1,3)/3 ones(1,3)/3 zeros(1,3)
5. Test for a single regressor in main effect of **B** (e.g. B2)
   0.5 0.5 0 1 0 0 0.5 0 0 0.5 0
6. Test for a single regressor in interaction **A** x **B** (e.g. A1B3)
   1 0 0 0 1 0 0 1 0 0 0

$$y_1 = \beta_1 + \beta_3 + \beta_6 + \varepsilon^{(1)}$$

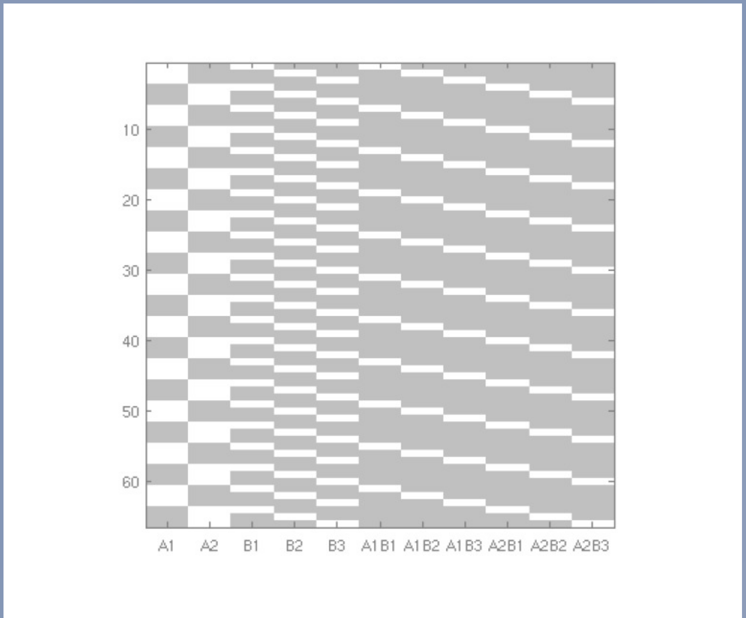
$$y_2 = \beta_1 + \beta_4 + \beta_7 + \varepsilon^{(2)}$$

$$y_3 = \beta_1 + \beta_5 + \beta_8 + \varepsilon^{(3)}$$

$$y_4 = \beta_2 + \beta_3 + \beta_9 + \varepsilon^{(4)}$$

$$y_5 = \beta_2 + \beta_4 + \beta_{10} + \varepsilon^{(5)}$$

$$y_6 = \beta_2 + \beta_5 + \beta_{11} + \varepsilon^{(6)}$$



$$y_1 + y_2 + y_3 = 3\beta_1 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 + \varepsilon^{1+2+3}$$

$$y_4 + y_5 + y_6 = 3\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_9 + \beta_{10} + \beta_{11} + \varepsilon^{4+5+6}$$

How is this computed ? (t-test)

Estimation [Y, X] [b, s]

$Y = X\beta + \varepsilon$ $\varepsilon \sim \sigma^2 N(0,I)$ *(Y : at one position)*

$b = (X'X)^+ X'Y$ *(b: estimate of β) -> beta??? images*

$e = Y - Xb$ *(e: estimate of ε)*

$s^2 = (e'e/(n - p))$ *(s: estimate of σ, n: time points, p: parameters)* *-> 1 image ResMS*

Test [b, s², c] [c'b, t]

$\mathrm{Var}(c'b) = s^2 c'(X'X)^+ c$ *(compute for each contrast c, proportional to s^2)*

$t = c'b / \mathrm{sqrt}(s^2 c'(X'X)^+ c)$ *c'b -> images spm_con???*
compute the t images -> images spm_t???

under the null hypothesis H_0 : $t \sim$ Student-t(df) $df = n-p$

How is this computed ? (F-test)

additional variance accounted for by tested effects
———
Error variance estimate

Estimation [Y, X] [b, s]

$$Y = X\beta + \varepsilon \qquad \varepsilon \sim N(0, \sigma^2 I)$$

$$Y = X_0 \beta_0 + \varepsilon_0 \qquad \varepsilon_0 \sim N(0, \sigma_0^2 I) \qquad X_0 : X\ Reduced$$

Test [b, s, c] [ess, F]

$$F \sim (s_0 - s) / s^2$$

-> image spm_ess???

-> image of F : spm_F???

under the null hypothesis : $F \sim F(p - p0, n-p)$