Bayesian Inference for MEG Source Reconstruction

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Abstract

1 Introduction

Given d MEG sensors and data from N time points, the MEG source reconstruction problem requires the inversion of the linear model

$$\begin{aligned} V &= XW + Z \\ (d \times N) &= [d \times p][p \times N] + [d \times N] \end{aligned}$$

where V is data matrix in sensor space, X is the lead field describing how neuronal currents from p cortical sources produce d sensor measurements, W is source activity at p source locations and N time points, and Z is a matrix of erros at d sensors and T time points.

1.1 Spatial Projector

To reduce the dimensionality of the problem one can project the data onto a *spatial projector* matrix U of dimension $[\tilde{d} \times d]$. We can then premultiply the above equation by U to give

$$\begin{aligned} \tilde{Y} &= LW + \tilde{Z} \\ (\tilde{d} \times N) &= [\tilde{d} \times p][p \times N] + [\tilde{d} \times N] \end{aligned}$$

where

$$\begin{aligned} \tilde{Y} &= UV \\ L &= UX \\ \tilde{Z} &= UZ \end{aligned}$$

For example, we may originally have $d = 274$ sensors but this can be reduced to $\tilde{d} = 87$ spatial modes. The *reduced* lead field is given by L.

1.2 Temporal Projector

To further reduce the dimensionality of the problem one can project the data onto a *temporal projector* matrix T of dimension $[\tilde{N} \times N]$. We can postmultiply the previous equation by T to give

$$\begin{aligned} Y &= LJ + E \\ (\tilde{d} \times \tilde{N}) &= [\tilde{d} \times p][p \times N] + [\tilde{d} \times \tilde{N}] \end{aligned}$$

where

$$\begin{aligned} J &= WT \\ E &= \tilde{Z}T \end{aligned}$$

For example, we may originally have $N = 161$ time points but this can be reduced to $\tilde{N} = 2$ temporal modes.

1.3 Bayesian Inversion

We first define the source and sensor space covariance matrices

$$\begin{aligned} C_j &= \mathsf{Cov}(J) \\ C_e &= \mathsf{Cov}(E) \end{aligned}$$

where C_e is $\tilde{d} \times \tilde{d}$ and C_j is $p \times p$.

The posterior distribution over sources is then given by

$$\begin{aligned} p(J|Y) &= \mathsf{N}(J; m, S) \\ S^{-1} &= L^T C_e^{-1} L + C_j^{-1} \\ m &= S L^T C_e^{-1} y \end{aligned}$$

The dimension of S is $p \times p$. This matrix is too large to invert so we can re-arrange the equations using the matrix inversion lemma.

1.4 Matrix Inversion Lemma

Otherwise known as the Woodbury identity this is

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B\left(C^{-1} + DA^{-1}B\right)^{-1} DA^{-1} \tag{1}$$

Applying this to the posterior covariance gives

$$S = C_j - C_j L^T (C_e + L C_j L^T)^{-1} L C_j$$

where the matrix inversion is now over a $\tilde{d} \times \tilde{d}$ matrix. If we define

$$V = (C_e + L C_j L^T)^{-1} \tag{2}$$

then we can write

$$S = C_j - C_j L^T V L C_j \tag{3}$$

or

$$S = C_j (I_p - L^T V L C_j) \tag{4}$$

1.5 Data projector

To compute the posterior mean (which is also the MAP estimator) we have

$$m = My \tag{5}$$

where

$$M = SL^T C_e^{-1} \tag{6}$$

By substituting in the previous expression for S we get

$$M = C_j(I_p - L^T V L C_j) L^T C_e^{-1} \tag{7}$$

1.6 Posterior variance

The posterior variance of the kth source is given by the kth diagonal entry in the posterior covariance matrix

$$\sigma_k^2 = S_{kk} \tag{8}$$

where

$$S = C_j - C_j L^T V L C_j \tag{9}$$

If we let

$$\Phi = LC_j \tag{10}$$

and denote the kth column of Φ as ϕ_k then we have

$$\sigma_k^2 = C_j(k,k) - \phi_k^T V \phi_k \tag{11}$$

This variance can be computed in a loop, $k = 1..p$, or perhaps more efficiently using a sparse matrix implementation for the second term.